FROBENIUS-UNSTABLE BUNDLES AND p-CURVATURE

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Abstract. We use the theory of p-curvature of connections to analyze stable vector bundles of rank 2 on curves of genus 2 which pull back to unstable bundles under the Frobenius morphism. We take two approaches, first using explicit formulas for p-curvature to analyze low-characteristic cases, and then using degeneration techniques to obtain an answer for a general curve by degenerating to an irreducible rational nodal curve, and applying the results of [13] and [15]. We also apply our explicit formulas to give a new description of the strata of curves of genus 2 of different p-rank.

1. Introduction

The primary theme of this paper is to use the following question as an invitation to a detailed study of the theory of p-curvature of connections:

Question 1.1. Given a smooth curve C of genus 2 over an algebraically closed field k of positive characteristic, what is the number of **Frobenius-unstable** vector bundles of rank 2 and trivial determinant on $C^{(p)}$? That is, if $F : C \to C^{(p)}$ denotes the relative Frobenius morphism from C to its p-twist, how many vector bundles $\mathscr{F}$ are there on $C^{(p)}$ (of the stated rank and determinant) which are themselves semistable, but for which $F^*\mathscr{F}$ is unstable?

Because semistability is preserved by pullback under separable morphisms (see [4, Lem. 3.2.2]), the Frobenius-unstable case is in some sense a universal case for destabilization. Furthermore, Frobenius-unstable bundles are closely related to the study of the generalized Verschiebung, and its relationship to p-adic representations of the fundamental group of C, in the case that C is defined over a finite field; see [12] for details.

The analysis of our question is in two parts: first, we use explicit formulas for p-curvature to calculate the answer directly for odd characteristics ≤ 7; and second, we use the abstract theory of p-curvature to give a new proof of the answer for a general curve of genus 2 in any odd characteristic, via degeneration to an irreducible rational nodal curve and application of the results of [13] and [15]. The latter result is originally due to Mochizuki; see [11] and [14]. The main advantage of the explicit approach, as compared to the more general degeneration argument, is that the p-curvature formulas may be used to study arbitrary smooth curves, and do not give results only for general curves. This distinction is underscored by an algorithm derived via the same techniques to explicitly describe the loci of curves of genus 2 and p-ranks 0 or 1 in any specified characteristic. Additionally, the explicit approach is useful for computing examples in order to formulate conjectures; one aim of this paper is therefore to serve as an illustration of how p-curvature may

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be used very concretely for experimental purposes, and more theoretically for more general results.

Our main theorem is:

Theorem 1.2. *Let C be a smooth, proper curve of genus 2 over an algebraically closed field k of characteristic p; it may be described on an affine part by $y^2 = g(x)$ for some quintic g. Then the number of semistable vector bundles on C with trivial determinant which pull back to unstable vector bundles under the relative Frobenius morphism is:*

- $p = 3$: $16 \cdot 1$;
- $p = 5$: $16 \cdot e_5$, *where $e_5 = 5$ for C general, and is given for an arbitrary C as the number of distinct roots of a quintic polynomial with coefficients in terms of the coefficients of g;*
- $p = 7$: $16 \cdot e_7$, *where $e_7 = 14$ for C general, and is given for an arbitrary C as the number of points in the intersection of four curves in $\mathbb{A}^2$ whose coefficients are in terms of the coefficients of g.*
- $p > 2$: *(Mochizuki [11], [14]) $16 \cdot \frac{p^3-p}{24}$ for C general.*

Furthermore, when C is general, any Frobenius-unstable bundle $\mathscr{F}$ has no nontrivial deformations which yield the trivial deformation of $F^*\mathscr{F}$.

There is a considerable amount of literature on Frobenius-unstable vector bundles. Gieseker and Raynaud produced certain examples of Frobenius-unstable bundles in [2] and [16, p. 119], but, aside from the results of Mochizuki discussed below, the first classification-type result is due to Laszlo and Pauly, who answered our main question in characteristic 2: there is always a single Frobenius-unstable bundle (see [9], argument for Prop. 6.1 2.; the equations for an ordinary curve are not used). Joshi, Ramanan, Xia and Yu obtain results on the Frobenius-unstable locus in characteristic 2 for higher-genus curves in [5]. Most recently, and concurrently with the initial preparation of the present paper, Lange and Pauly [8] have recovered the formula of Theorem 1.2 for general C in the case of ordinary curves via a completely different approach, although they obtain only an inequality, rather than an equality.

However, the most comprehensive results to date follow from Mochizuki's work (see [11] and [14]), which was carried out in the context of $\mathbb{P}^1$-bundles on curves in any odd characteristic, via degeneration techniques quite similar to those which we pursue in Sections 8 and 9. Indeed, key results and their arguments in Sections 7, 8, and 9 are essentially the same as Mochizuki's; in the first case, the argument presented here was discovered independently, while in the other cases, the author's original arguments were more complicated and less general than Mochizuki's, and have thus been replaced. There are several justifications for the logical redundancy: the arguments in question are all quite short, and it seems desirable to have a self-contained proof of the main theorem, without translating to projective bundles and back; the argument of Section 7 is actually substantially simpler in our case of curves of genus 2; and finally, the gluing statements of Section 8 require some ridigifying hypotheses in the context of vector bundles that do not arise in Mochizuki's work.

Lastly, we remark that as discussed in [14], Mochizuki's strategy is to degenerate to totally degenerate curves, while our strategy is to degenerate to irreducible nodal curves. Aside from allowing one to make more naive arguments in terms of explicit degenerations, ours is a substantially more difficult approach, since after reducing

the problem to self-maps of $\mathbb{P}^1$ with prescribed ramification, in Mochizuki's case it suffices to handle the case of three ramification points, while our argument requires four, and is therefore far more complicated; see [15] for details. However, degenerating to irreducible curves is helpful for studying Frobenius-unstable bundles in higher genus; see [14].

We begin in Section 2 by relating our main question to p-curvature, and Section 3 is then devoted to developing explicit and completely general combinatorial formulas for p-curvature. We make certain necessary computations for genus 2 curves in Section 4, which we also apply to obtain an explicit algorithm for generating p-rank formulas in any given odd characteristic. Section 5 is devoting to computing the space of connections on a certain unstable bundle, and in Section 6 we conclude the computation with explicit descriptions of the locus of vanishing p-curvature in characteristics $3, 5$ and 7. The space of connections on the same bundle having nilpotent p-curvature is shown to be finite and flat in Section 7, again by explicit computation; this completes the proof of Theorem 1.2 for $p \leq 7$, and also provides a key step of the general case. In Section 8 we discuss the relationship between connections on nodal curves and their normalizations, and finally in Section 9 we show that connections on nodal curves deform, and apply the results of [13] and [15] to conclude our main theorem.

Computations were carried out in Maple and Mathematica, and in the case of the p-curvature formulas of Section 3, using simple C code.

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2. From Frobenius-instability to p-curvature

We begin by explaining how classification of Frobenius-unstable vector bundles is related to p-curvature of connections. For the basic theory of connections and p-curvature, we refer the reader to [7, §1, §5]. Note that the induced connection on tensor products descends to wedge products, so that for a vector bundle $\mathscr{E}$ with connection, we obtain an induced **determinant connection** on $\det \mathscr{E}$. Additionally, given $\varphi \in \operatorname{Aut}(\mathscr{E})$ and a ∇ on $\mathscr{E}$, we refer to the operation of conjugation by φ on ∇ as **transport**. We summarize the basic results relating Frobenius with p-curvature, due to to Katz [7].

Theorem 2.1. *Let X be a smooth S-scheme, with S having characteristic p, and let $F : X \to X^{(p)}$ be the relative Frobenius morphism. Then for any vector bundle $\mathscr{F}$ on $X^{(p)}$, $F^*\mathscr{F}$ is equipped with a canonical connection ∇^{can}. For any vector bundle $\mathscr{E}$ with connection ∇ on X, the kernel of ∇, denoted $\mathscr{E}^{\nabla}$, is naturally an $\mathscr{O}_{X^{(p)}}$-module.*

The operations $\mathscr{F} \mapsto (F^*\mathscr{F}, \nabla^{can})$ and $(\mathscr{E}, \nabla) \mapsto \mathscr{E}^{\nabla}$, are mutually inverse functors, giving an equivalence of categories between the category of vector bundles of rank n on $X^{(p)}$ and the full subcategory of the category of vector bundles of rank n with integrable connection on X consisting of objects whose connection has p-curvature zero.

Furthermore, the same statement holds when restricted to the full subcategories of vector bundles with trivial determinant on $X^{(p)}$, and vector bundles with connection both having trivial determinant on X.

Proof. See [7, §5], and in particular [7, Thm. 5.1]. It only remains to check that the categorical equivalence on coherent sheaves gives an equivalence on vector bundles, and again in the case of trivial determinant. The first assertion follows from the fact that F is faithfully flat when X/S is smooth. The second is easily checked by verifying that the operation $\mathscr{F} \mapsto (F^*\mathscr{F}, \nabla^{\mathrm{can}})$ commutes with taking determinants. □

Thus, p-curvature is naturally related to the study of Frobenius-pullbacks. The categorical equivalence implies that isomorphism classes of $\mathscr{F}$ will correspond to transport equivalence classes of connections with vanishing p-curvature. In the case of our particular question, the relationship is particularly helpful. We assume we are in the following situation.

Situation 2.2. C is a smooth, proper curve of genus 2, over an algebraically closed field k of characteristic p.

In this situation, Joshi and Xia showed that there are at most finitely many Frobenius-unstable vector bundles of rank 2 and trivial determinant on C (see [6, Thm. 3.2], although we will also obtain a more direct proof from Corollary 7.2), and also gave the following description of them (see [6, Prop. 3.3]):

Proposition 2.3. *(Joshi-Xia) Let $\mathscr{F}$ be a semistable rank 2 vector bundle on C with trivial determinant, and suppose $\mathscr{E} = F^*\mathscr{F}$ is unstable. Then there is a non-split exact sequence*

$$0 \to \mathscr{L} \to \mathscr{E} \to \mathscr{L}^{-1} \to 0$$

where $\mathscr{L}$ is a **theta characteristic***, that is, $\mathscr{L}^{\otimes 2} \cong \Omega^1_C$.*

We thus have a natural set of unstable vector bundles upon which to look for connections with vanishing p-curvature. Indeed, it is easy to see that the proposition is sharp.

Corollary 2.4. *Frobenius-unstable vector bundles of rank 2 and trivial determinant on C are necessarily stable, and in one-to-one correspondence with transport-equivalence classes of connections on vector bundles $\mathscr{E}$ as in the above proposition, having trivial determinant and vanishing p-curvature. This correspondence is functorial in the sense that after arbitrary base change $C' \to C$, vector bundles $\mathscr{F}$ with trivial determinant and $F^*\mathscr{F} \cong \mathscr{E}'$ are in one-to-one correspondence with transport-equivalence classes of connections on $\mathscr{E}'$ having trivial determinant and vanishing p-curvature.*

Proof. The functoriality is the more obvious statement, in light Theorem 2.1. For the rest, all we need check is that if $F^*\mathscr{F} \cong \mathscr{E}$ for some $\mathscr{F}$, we necessarily have that $\mathscr{F}$ is stable. But if $\mathscr{M} \subset \mathscr{F}$ is a non-negative line sub-bundle, $F^*\mathscr{M} \subset \mathscr{F}$ is non-negative with degree a multiple of p, which cannot occur when $F^*\mathscr{F} \cong \mathscr{E}$ by the following standard lemma. □

We state the lemma in more generality than immediately necessary, for later use. The argument for the $\mathcal{E}nd^0(\mathscr{E})$ case is taken from [11, Lem. I.3.5, p. 105].

Lemma 2.5. *Let $\mathscr{E}$ be a rank 2 vector bundle of degree 0 on a possibly nodal curve C, and suppose $\mathscr{L}$ is a positive line bundle giving an exact sequence*

$$0 \to \mathscr{L} \to \mathscr{E} \to \mathscr{E}/\mathscr{L} \to 0$$

Then $\mathscr{L}$ is unique, and is the maximal degree line bundle inside $\mathscr{E}$, and $\mathscr{E}$ has no quotient line bundle of degree 0. Furthermore, the same statement holds for positive sub-bundles of given rank of the traceless endomorphisms $\mathcal{E}nd^0(\mathscr{E})$.

Proof. One checks this simply by considering maps of the form $\mathscr{L} \to \mathscr{E} \to \mathscr{E}/\mathscr{L}'$, and considering the degrees of the line bundles in question. For the $\mathcal{E}nd^0(\mathscr{E})$ case, because $\mathcal{E}nd^0(\mathscr{E})$ is self-dual it suffices to consider the case of line sub-bundles, and to show that the existence of a positive sub-bundle precludes the existence of a line sub-bundle of degree 0. But if we have $\mathscr{L} \subset \mathcal{E}nd^0(\mathscr{E})$ positive, and $\mathscr{L}' \subset \mathcal{E}nd^0(\mathscr{E})$ non-negative, first by considering $\mathscr{L}' \to \mathcal{E}nd^0(\mathscr{E}) \to \mathscr{L}^{-1}$ we find that the composition must be zero, so that we have a map $\mathcal{E}nd^0(\mathscr{E})/\mathscr{L}' \to \mathscr{L}^{-1}$. But then considering the natural $\mathscr{O}_C \subset \mathcal{E}nd^0(\mathscr{E})/\mathscr{L}'$, composing with the map to $\mathscr{L}^{-1}$ must again give zero, so that in fact the map $\mathcal{E}nd^0(\mathscr{E}) \to \mathscr{L}^{-1}$ factors through $(\mathcal{E}nd^0(\mathscr{E})/\mathscr{L}')/\mathscr{O}_C \cong \mathscr{L}'^{-1}$, from which one can conclude the desired statement. □

Next, we note that the $\mathscr{E}$ of Proposition 2.3 are nearly unique.

Proposition 2.6. *There are only 16 choices for $\mathscr{E}$ as described in Proposition 2.3, one for each choice of $\mathscr{L}$.*

Proof. Any two choices of $\mathscr{L}$ differ by one of the $2^{2g} = 16$ line bundle of order 2 on C. With $\mathscr{L}$ chosen, we calculate that $\mathrm{Ext}^1(\mathscr{L}^{-1}, \mathscr{L}) \cong H^0(C, \mathscr{O}_C) \cong k$, so the isomorphism class of $\mathscr{E}$ is uniquely determined. □

Lastly, we observe that it suffices to handle a single choice of $\mathscr{E}$.

Corollary 2.7. *For any $\mathscr{E}, \mathscr{E}'$ as in Proposition 2.6, there is a canonical funtorial equivalence between the vector bundles $\mathscr{F}$ of trivial determinant with $F^*\mathscr{F} \cong \mathscr{E}$, and those with $F^*\mathscr{F} \cong \mathscr{E}'$.*

Proof. From Proposition 2.6 we see that $\mathscr{E}$ and $\mathscr{E}'$ are related by tensoring by a 2-torsion line bundle. The corollary is then easily verified by the bijectivity of F^* on 2-torsion line bundles. □

Having reduced our main question to a matter of classifying connections with vanishing p-curvature on a certain vector bundle, we briefly develop the formal properties of p-curvature, which we will not need to use until Section 7 and the following sections. The statement is:

Proposition 2.8. *Given a connection ∇ on a vector bundle $\mathscr{E}$ on a smooth X over S, we have the following description of the p-curvature ψ_∇ of ∇.*

(i) *We may describe ψ_∇ as an element of*

$$\Gamma(X, \mathcal{E}nd(\mathscr{E}) \otimes F^*\Omega^1_{X^{(p)}/S})^{\nabla^{\mathrm{ind}}},$$

where the superscript denotes the subspace of sections horizontal for ∇^{ind};

(ii) *If $\mathscr{E}$ and ∇ have trivial determinant, we find that ψ_∇ lies in*

$$\Gamma(X, \mathcal{E}nd^0(\mathscr{E}) \otimes F^*\Omega^1_{X^{(p)}/S})^{\nabla^{\mathrm{ind}}},$$

where $\mathcal{E}nd^0(\mathscr{E})$ denotes the sheaf of traceless endomorphisms of $\mathscr{E}$.

(iii) *Assuming $\mathscr{E}$ has a connection, we may also consider p-curvature as giving maps between affine spaces*

$$\psi : \Gamma(X, \mathrm{Conn}(\mathscr{E})) \to \Gamma(X, \mathcal{E}nd(\mathscr{E}) \otimes F^*\Omega^1_{X^{(p)}}),$$

$$\psi^0 : \Gamma(X, \mathrm{Conn}^0(\mathscr{E})) \to \Gamma(X, \mathcal{E}nd^0(\mathscr{E}) \otimes F^*\Omega^1_{X^{(p)}}),$$

where $\mathrm{Conn}(\mathscr{E})$ *and* $\mathrm{Conn}^0(\mathscr{E})$ *denote the sheaves of connections on $\mathscr{E}$, and of connections with trivial determinant (when $\mathscr{E}$ has trivialized determinant) respectively.*

(iv) *We may take the determinant of the previous maps, and in the case that $\mathscr{E}$ has trivial determinant, we obtain a map*

$$\det \psi^0 : \Gamma(X, \mathrm{Conn}^0(\mathscr{E})) \to \Gamma(X^{(p)}, (\Omega^1_{X^{(p)}})^{\otimes n}).$$

Proof. Assertion (i) follows directly from the linearity and p-linearity results of Katz [7, 5.0.5, 5.2.0], together with the fact that $\psi_\nabla(\theta)$ commutes with $\nabla_{\theta'}$ for any θ', by[7, 5.2.3]. Assertion (ii) follows from explicit computation, in Corollary 3.6 (ii). We then obtain assertion (iii) formally: since we are working over an arbitrary scheme, we obtain the map on arbitrary T-valued points, and if $\mathscr{E}$ has a connection, the space of connections is a torsor over $\Gamma(X, \mathscr{E} \otimes \Omega^1_X)$, and likewise after arbitrary pull-back, and hence representable by an affine space. Finally, for assertion (iv), we just put together assertions (ii) and (iii), checking that in the trivial determinant case, the induced connection on the determinant of $\mathrm{End}^0(\mathscr{E})$ is likewise trivial. □

3. Explicit p-curvature Formulas

In this section, we develop general combinatorial formulas which may be used to explicitly compute the p-curvature of a connection for any given p, and in any dimension, although it will be easiest to compute in the case of curves, where it suffices to consider a single derivation. We specify our notation for the section.

Situation 3.1. U denotes an affine open on a smooth X/S. We are given a vector bundle $\mathscr{E}$ trivialized on U, and a derivation θ on U. We thus obtain a connection matrix $\bar{T}$ on U associated to any connection ∇ on $\mathscr{E}$, such that $\nabla_\theta(s) = \bar{T}s + \theta s$. Denote also by $\bar{T}_{(p)}$ the connection matrix associated to ∇ and θ^p.

One can then easily check the following explicit formula for the p-curvature associated to ∇ and θ.

Lemma 3.2. *We have* $\psi_\nabla(\theta) = (\bar{T} + \theta)^p - \bar{T}_{(p)} - \theta^p$

We now describe the expansion of $(\bar{T} + \theta)^n$ using the commutation relation $\theta\bar{T} = (\theta\bar{T}) + \bar{T}\theta$, where, in order to make formulas easier to parse, $(\theta\bar{T})$ denotes the application of θ to the coordinates of $\bar{T}$.

Proposition 3.3. *Given* $\mathfrak{i} = (i_1, \dots, i_\ell) \in \mathbb{N}^{\ell-1} \times (\mathbb{N} \cup \{0\})$ *with* $\sum_{j=1}^{\ell} i_j = n$, *denote by* $\hat{n}_\mathfrak{i}$ *the coefficient of* $\bar{T}_\mathfrak{i} := (\theta^{i_1-1}\bar{T}) \dots (\theta^{i_{\ell-1}-1}\bar{T})\theta^{i_\ell}$ *in the full expansion of* $(\bar{T} + \theta)^n$. *Also denote by* $\mathfrak{i}_0$ *the vector* $(i_1, \dots, i_{\ell-1}, 0)$. *Then we have:*

$$\hat{n}_\mathfrak{i} = \binom{n}{i_\ell} \hat{n}_{\mathfrak{i}_0}$$

Proof. Although this formula may be seen directly, the proof is expressed most clearly by induction on n, which we sketch. We may assume that $i_\ell > 0$, or the statement is trivial. By definition, we have

$$(\bar{T}+\theta)^n = (\bar{T}+\theta)(\sum_{\ell'} \sum_{|\mathfrak{i}'|=n-1} \hat{n}_{\mathfrak{i}'}\bar{T}_{\mathfrak{i}'}),$$

where $\mathfrak{i}' = (i'_1, \ldots, i'_{\ell'})$, and $|\mathfrak{i}'| := \sum_j i'_j$. Multiplying out and commuting the θ from left to right until we obtain another such expression, we find two cases: $i_1 = 1$ and $i_1 > 1$; we handle the case $i_1 > 1$, the other being essentially the same. In this case, we obtain the inductive formula $\hat{n}_\mathfrak{i} = \sum_j \hat{n}_{\mathfrak{i}-1_j}$, where 1_j denotes the vector which is 1 in the jth position and 0 elsewhere, and where j is allowed to range only over values where $i_j > 1$. We then have also that $\hat{n}_{\mathfrak{i}_0} = \sum_{j<\ell} \hat{n}_{\mathfrak{i}_0-1_j}$, so that if we induct on n, we have $\hat{n}_i = \sum_j \hat{n}_{\mathfrak{i}-1_j} = \sum_{j<\ell} \binom{n-1}{i_\ell}\hat{n}_{(\mathfrak{i}-1_j)_0} + \binom{n-1}{i_{\ell}-1}\hat{n}_{(\mathfrak{i}-1_\ell)_0} = (\binom{n-1}{i_\ell} + \binom{n-1}{i_\ell-1})\hat{n}_{\mathfrak{i}_0}$, where the last equality makes use of the observation that $\mathfrak{i}_0 = (\mathfrak{i}-1_\ell)_0$. Then the identity $\binom{n-1}{r} + \binom{n-1}{r-1} = \binom{n}{r}$ completes the proof. □

It follows that if $n = p$, $\hat{n}_\mathfrak{i}$ is nonzero mod p only if $i_\ell = 0$ or $i_\ell = p$, and in the latter case, we have $\ell = 1$, $i_1 = p$, and $\hat{n}_\mathfrak{i} = 1$, which precisely cancels the θ^p subtracted off in the formula for $\psi_\nabla(\theta)$. We immediately see that $\psi_\nabla(\theta)$ is in fact given entirely by linear terms. In particular, this explicitly recovers the statement we already knew to be true that p-curvature takes values in the space of $\mathscr{O}_C$-linear endomorphisms of $\mathscr{E}$. We may now restrict our attention to the linear terms in the expansion, and will shift our notation accordingly:

Proposition 3.4. *Given $\mathfrak{i} = (i_1, \ldots, i_\ell) \in \mathbb{N}^\ell$ with $\sum_{j=1}^{\ell} i_j = n$, denote by $n_\mathfrak{i}$ the coefficient of $\bar{T}_\mathfrak{i} = (\theta^{i_1-1}\bar{T})\ldots(\theta^{i_\ell-1}\bar{T})$ in the full expansion of $(\bar{T}+\theta)^n$. Also denote by $\hat{\mathfrak{i}}$ the truncated vector $(i_1, \ldots, i_{\ell-1})$. Then we have:*

$$n_\mathfrak{i} = \binom{n-1}{i_\ell - 1} n_{\hat{\mathfrak{i}}}$$

We thus get

$$n_\mathfrak{i} = \prod_{j=1}^{\ell} \binom{n-1-\sum_{m=j+1}^{\ell} i_m}{i_j - 1} = \frac{(n-1)!}{(\prod_{j=1}^{\ell}(i_j-1)!)(\prod_{j=1}^{\ell-1}(\sum_{m=1}^{j} i_m))}$$

Proof. This follows from the same induction argument as the previous proposition. □

We note that this implies that every such term in the expansion of $(\bar{T}+\theta)^n$ is nonzero mod n when $n = p$, since the numerator in the resulting formula is simply $(n-1)!$. Thus, the p-curvature formula is always maximally complex, having an exponential number of terms. However, when some of the terms commute, the formulas tend to simplify considerably.

Proposition 3.5. *Given $\ell > 0$ and a subset $\Lambda \subset \{1, \ldots, \ell\}$, denote by S_ℓ^Λ the subset of the permutation group S_ℓ which preserves the order of the elements of Λ; that is, $S_\ell^\Lambda := \{\sigma \in S_\ell : \forall j < j' \in \Lambda, \sigma(j) < \sigma(j')\}$. Given also $\mathfrak{i} = (i_1, \ldots, i_\ell) \in \mathbb{N}^\ell$ with $\sum_{j=1}^{\ell} i_j = n$, we denote by $n_\mathfrak{i}^\Lambda$ the sum over all $\sigma \in S_\ell^\Lambda$ of $n_{\sigma(\mathfrak{i})}$, where $\sigma(\mathfrak{i})$*

denotes the vector $(i_{\sigma^{-1}(1)},\dots,i_{\sigma^{-1}(\ell)})$ *obtained from* $\mathfrak{i}$ *by permuting the coordinates under* σ*. Then we have:*

$$n_{\mathfrak{i}}^{\Sigma}=\frac{n!}{\prod_{j=1}^{\ell}(i_j-1)!\prod_{j=1}^{\ell}(i_j+\sum_{m<j}^{m,j\in\Lambda}i_m)}.$$

Note that the last sum in the denominator is non-empty only for $j\in\Lambda$*.*

Proof. First note that if we want the entries of $\mathfrak{i}$ with indices in Λ to have the same order in $\sigma(\mathfrak{i})$, we must apply σ^{-1} rather than σ to the indices, as in our definition. Applying our previous formula, we really just want to show that

$$\sum_{\sigma\in S_\ell^\Lambda}\prod_{j=1}^{\ell-1}\frac{1}{\sum_{m=1}^{j}i_{\sigma^{-1}(m)}}=\frac{n}{\prod_{j=1}^{\ell}(i_j+\sum_{m<j}^{m,j\in\Lambda}i_m)}=\frac{\sum_{j=1}^{\ell}i_j}{\prod_{j=1}^{\ell}(i_j+\sum_{m<j}^{m,j\in\Lambda}i_m)}.$$

Dividing through by $\sum_{j=1}^{\ell}i_j$ reduces the identity to

$$\sum_{\sigma\in S_\ell^\Lambda}\prod_{j=1}^{\ell}\frac{1}{\sum_{m=1}^{j}i_{\sigma^{-1}(m)}}=\frac{1}{\prod_{j=1}^{\ell}(i_j+\sum_{m<j}^{m,j\in\Lambda}i_m)}. \tag{3.1}$$

We show this by induction on ℓ (noting that it is rather trivial in the case $\ell=1$, whether or not Λ is empty), breaking up the first sum over S_ℓ^Λ into $\ell-|\Lambda|+1$ pieces, depending on which i_r ends up in the final place. There are two cases to consider: $r\notin\Lambda$, or $r=\Lambda_{\max}$. In either case, the relevant part of the sum on the left hand side becomes $\sum_{\sigma\in S_\ell^{\Lambda,r}}\prod_{j=1}^{\ell}\frac{1}{\sum_{m=1}^{j}i_{\sigma^{-1}(m)}}$, where $S_\ell^{\Lambda,r}$ denotes the subset of S_ℓ^Λ sending r to ℓ. Now, the point is that for our sums, this will be equivalent to an order-preserving subset of the symmetric group acting on a set of $\ell-1$ elements, allowing us to apply induction. In the case that $r\notin\Lambda$, Λ is in essence unaffected, and we find that

$$\sum_{\sigma\in S_\ell^{\Lambda,r}}\prod_{j=1}^{\ell}\frac{1}{\sum_{m=1}^{j}i_{\sigma^{-1}(m)}}=\frac{1}{n}\sum_{\sigma\in S_\ell^{\Lambda,r}}\prod_{j=1}^{\ell-1}\frac{1}{\sum_{m=1}^{j}i_{\sigma^{-1}(m)}},$$

and one checks that this sum is of the same form as Equation 3.1, with i_r ommitted, so by induction we find that this sum is equal to

$$\frac{1}{n}\frac{1}{\prod_{j\neq r}(i_j+\sum_{m<j}^{m,j\in\Lambda}i_m)}=\frac{i_r+\sum_{m<r}^{m,r\in\Lambda}i_m}{n\prod_{j=1}^{\ell}(i_j+\sum_{m<j}^{m,j\in\Lambda}i_m)}=\frac{i_r}{n\prod_{j=1}^{\ell}(i_j+\sum_{m<j}^{m,j\in\Lambda}i_m)},$$

since $r\notin\Lambda$. In the case that $r=\Lambda_{\max}$, we effectively reduce the size of Λ by one, but because r is maximal in Λ, for $j\neq r$ the term $\sum_{m<j}^{m,j\in\Lambda}i_m$ is unaffected by omitting r from Λ. We thus find, arguing as before,

$$\sum_{\sigma\in S_\ell^{\Lambda,r}}\prod_{j=1}^{\ell}\frac{1}{\sum_{m=1}^{j}i_{\sigma^{-1}(m)}}=\frac{i_r+\sum_{m<r}^{m,r\in\Lambda}i_m}{n\prod_{j=1}^{\ell}(i_j+\sum_{m<j}^{m,j\in\Lambda}i_m)}=\frac{\sum_{j\in\Lambda}i_j}{n\prod_{j=1}^{\ell}(i_j+\sum_{m<j}^{m,j\in\Lambda}i_m)}.$$

Adding these up as r ranges over $\Lambda_{\max}$ and all values not in Λ, and using $n=\sum_j i_j$, we get the desired identity. $\square$

We give some specific applications of this formula.

Corollary 3.6. *Let $\mathscr{E}$ be a vector bundle of rank r on a smooth variety X over a field k, with ∇ an integrable connection on $\mathscr{E}$ and θ a derivation on an open set U which also trivializes $\mathscr{E}$. We have:*

(i) *If $r = 1$, p-curvature is given by:*
$$\psi_\nabla(\theta) = \bar{T}^p + (\theta^{p-1}\bar{T}) - \bar{T}_{(p)}.$$

(ii) *Suppose $\mathscr{E}$ has trivialized determinant, and ∇ has trivial determinant. Then the p-curvature of ∇ has image in the traceless endomorphisms of $\mathscr{E}$.*

(iii) *Suppose ∇' is a connection on U with $\nabla' - \nabla = \omega\,\mathrm{I}$ a scalar endomorphism. Then we have*
$$\psi_{\nabla'}(\theta) - \psi_\nabla(\theta) = ((\hat{\theta}(\omega))^p + \theta^{p-1}(\hat{\theta}(\omega)) - \hat{\theta^p}(\omega))\,\mathrm{I},$$
where $\hat{\theta}$ denotes the unique linear map $\Omega^1_C \to \mathscr{O}_C$ such that $\theta = \hat{\theta} \circ d$.

Proof. From the previous proposition, we see that when $n = p$ and Λ is empty, so that all the involved matrices commute, we have
$$n_{\mathrm{i}}^{\varnothing} = \frac{p!}{\prod_{j=1}^{\ell} i_j!},$$
but the actual coefficient will be $n_{\mathrm{i}}^{\varnothing}/P_{\mathrm{i}}$, where P_{i} is the number of permutations fixing the vector i, since summing up over all permutations will count each term P_{i} times. We see that this expression can be non-zero mod p only if either P_{i} is a multiple of p, or some i_j is. Since P_{i} is the order of a subgroup of S_ℓ, it can be a multiple of p if and only if $\ell = p$ and each $i_j = 1$. On the other hand, an i_j can be a multiple of p if and only if $\ell = 1$ and $i_1 = p$; these two terms simply reiterate that the coefficients of $\bar{T}^p$ and $(\theta^{p-1}\bar{T})$ are both 1, and we see that every other coefficient vanishes mod p. We immediately conclude (i), and for (ii) we see similarly that we have
$$\operatorname{Tr}\psi_\nabla(\theta) = \operatorname{Tr}\bar{T}^p + \operatorname{Tr}(\theta^{p-1}\bar{T}) - \operatorname{Tr} f_{\theta^p}\bar{T}.$$
The second and third terms visibly have vanishing trace because $\bar{T}$ does, while it is easy to see (for instance, by passing to the algebraic closure of k and taking the Jordan normal form) that $\operatorname{Tr}(\bar{T}^p) = (\operatorname{Tr}\bar{T})^p = 0$.

For (iii), we can compare the p-curvatures of ∇ and ∇' term by term; we have $\bar{T} + \hat{\theta}(\omega)\,\mathrm{I}$ as the matrix for ∇', and we see that if we expand each term of $\psi_{\nabla'}(\theta)$, we get $\psi_\nabla(\theta)$ from expanding out only terms involving $\bar{T}$ and θ, and $((\hat{\theta}(\omega))^p + \theta^{p-1}(\hat{\theta}(\omega)) - \hat{\theta^p}(\omega))\,\mathrm{I}$ from expanding out terms involving only $\hat{\theta}(\omega)$ and θ, since these last all commute with one another. We thus want to show that all of the coefficients of the cross terms are always zero mod p. If we consider a particular term $(\theta_0^{i_1-1}(\bar{T} + \hat{\theta}(\omega)\,\mathrm{I}))\dots(\theta_0^{i_\ell-1}(\bar{T} + \hat{\theta}(\omega)\,\mathrm{I}))$ corresponding to a vector i, a cross term will arise by choosing a subset $\Lambda \subset \{1, \dots, \ell\}$ from which the $\bar{T}$ terms will be chosen, with the $\hat{\theta}(\omega)\,\mathrm{I}$ term being chosen for all indices outside Λ. To compute the relevant coefficient we can essentially sum over all permutations in the S_ℓ^Λ of Proposition 3.5. The only caveat is that if $\sigma \in S_\ell^\Lambda$ fixes Λ and leaves the vector i unchanged, then it will give the same term in the expansion as the identity. Such σ form a subgroup of S_ℓ, and if we denote the order of this subgroup by P_{i}^Λ, we find that the coefficient we want to compute is given by, still in the notation of Proposition 3.5, the expression $n_{\mathrm{i}}^\Lambda/P_{\mathrm{i}}^\Lambda$. Now, the only way to cancel the p in

the numerator of $n_{\rm i}^{\Lambda}$ would be for either $P_{\rm i}^{\Lambda}$ or the denominator of $n_{\rm i}^{\Lambda}$ to also be divisible by p. The denominator of $n_{\rm i}^{\Lambda}$ cannot be divisible by p, since the i_j add up to p, and the only way that p could appear in the denominator would therefore be when Λ is all of $\{1, \dots, \ell\}$, which corresponds to the terms which only involve $\bar{T}$, or when $\ell = 1$, which gives the $\theta^{p-1}(\hat{\theta}(\omega))$ term. Similarly, $P_{\rm i}^{\Lambda}$ is the order of a subgroup of S_ℓ which fixes Λ, so can be a multiple of p only if $\ell = p$ and $|\Lambda| = 0$, which corresponds to the term $((\hat{\theta}(\omega))^p$. This yields the desired result. □

We do not use the last statement of the corollary in this paper, but it could be used to show, for instance, that when r is prime to p, and $\mathscr{E}$ has rank r, then if any representative on $\mathscr{E}$ of a projective connection ∇ on $\mathbb{P}(\mathscr{E})$ has vanishing p-curvature, then the unique representative on $\mathscr{E}$ of ∇ with vanishing trace must likewise have vanishing p-curvature. We also remark that results such as statement (ii) above may generally be obtained more abstractly via general functoriality statements on p-curvature, but such a point of view requires familiarity with Grothendieck's abstract theory of connections; see [14].

We conclude with some observations in the case of curves.

Lemma 3.7. *In the case that X is a curve, the p-curvature of a connection ∇ is identically 0 if and only if $\psi_\nabla(\theta) = 0$ for any non-zero derivation θ. In addition, $\bar{T}_{(p)} = f_{\theta^p}\bar{T}$ for some function f_{θ^p}, satisfying $f_{\theta^p}\theta = \theta^p$.*

Proof. These statements follow trivially from the fact that the sheaf of derivations is invertible, and the p-linearity of the p-curvature map with respect to derivations. □

Finally, we record in this situation the general p-curvature formulas in characteristics 3, 5, and 7, for later use.

Characteristic 3:

$$\psi_\nabla(\theta) = \bar{T}^3 + (\theta\bar{T})\bar{T} + 2\bar{T}(\theta\bar{T}) + (\theta^2\bar{T}) - f_{\theta^3}\bar{T} \tag{3.2}$$

Characteristic 5:

$$\begin{aligned}\psi_\nabla(\theta) = \bar{T}^5 &+ 4\bar{T}^3(\theta^1\bar{T}) + 3\bar{T}^2(\theta^1\bar{T})\bar{T} + \bar{T}^2(\theta^2\bar{T}) + 2\bar{T}(\theta^1\bar{T})\bar{T}^2 \\ &+ 3\bar{T}(\theta^1\bar{T})^2 + 3\bar{T}(\theta^2\bar{T})\bar{T} + 4\bar{T}(\theta^3\bar{T}) + (\theta^1\bar{T})\bar{T}^3 \\ &+ 4(\theta^1\bar{T})\bar{T}(\theta^1\bar{T}) + 3(\theta^1\bar{T})^2\bar{T} + (\theta^1\bar{T})(\theta^2\bar{T}) + (\theta^2\bar{T})\bar{T}^2 \\ &+ 4(\theta^2\bar{T})(\theta^1\bar{T}) + (\theta^3\bar{T})\bar{T} + (\theta^4\bar{T}) - f_{\theta^5}\bar{T}\end{aligned} \tag{3.3}$$

Characteristic 7:

$$
\begin{aligned}
\psi_\nabla(\theta) = {} & \bar{T}^7 + 6\bar{T}^5(\theta^1\bar{T}) + 5\bar{T}^4(\theta^1\bar{T})\bar{T} + \bar{T}^4(\theta^2\bar{T}) + 4\bar{T}^3(\theta^1\bar{T})\bar{T}^2 \\
& + 3\bar{T}^3(\theta^1\bar{T})^2 + 3\bar{T}^3(\theta^2\bar{T})\bar{T} + 6\bar{T}^3(\theta^3\bar{T}) + 3\bar{T}^2(\theta^1\bar{T})\bar{T}^3 \\
& + 4\bar{T}^2(\theta^1\bar{T})\bar{T}(\theta^1\bar{T}) + \bar{T}^2(\theta^1\bar{T})^2\bar{T} + 3\bar{T}^2(\theta^1\bar{T})(\theta^2\bar{T}) + 6\bar{T}^2(\theta^2\bar{T})\bar{T}^2 \\
& + \bar{T}^2(\theta^2\bar{T})(\theta^1\bar{T}) + 3\bar{T}^2(\theta^3\bar{T})\bar{T} + \bar{T}^2(\theta^4\bar{T}) + 2\bar{T}(\theta^1\bar{T})\bar{T}^4 \\
& + 5\bar{T}(\theta^1\bar{T})\bar{T}^2(\theta^1\bar{T}) + 3\bar{T}(\theta^1\bar{T})\bar{T}(\theta^1\bar{T})\bar{T} + 2\bar{T}(\theta^1\bar{T})\bar{T}(\theta^2\bar{T}) \\
& + \bar{T}(\theta^1\bar{T})^2\bar{T}^2 + 6\bar{T}(\theta^1\bar{T})^3 + 6\bar{T}(\theta^1\bar{T})(\theta^2\bar{T})\bar{T} + 5\bar{T}(\theta^1\bar{T})(\theta^3\bar{T}) \\
& + 3\bar{T}(\theta^2\bar{T})\bar{T}^3 + 4\bar{T}(\theta^2\bar{T})\bar{T}(\theta^1\bar{T}) + \bar{T}(\theta^2\bar{T})(\theta^1\bar{T})\bar{T} + 3\bar{T}(\theta^2\bar{T})^2 \\
& + 4\bar{T}(\theta^3\bar{T})\bar{T}^2 + 3\bar{T}(\theta^3\bar{T})(\theta^1\bar{T}) + 5\bar{T}(\theta^4\bar{T})\bar{T} + 6\bar{T}(\theta^5\bar{T}) + (\theta^1\bar{T})\bar{T}^5 \\
& + 6(\theta^1\bar{T})\bar{T}^3(\theta^1\bar{T}) + 5(\theta^1\bar{T})\bar{T}^2(\theta^1\bar{T})\bar{T} + (\theta^1\bar{T})\bar{T}^2(\theta^2\bar{T}) \\
& + 4(\theta^1\bar{T})\bar{T}(\theta^1\bar{T})\bar{T}^2 + 3(\theta^1\bar{T})\bar{T}(\theta^1\bar{T})^2 + 3(\theta^1\bar{T})\bar{T}(\theta^2\bar{T})\bar{T} \\
& + 6(\theta^1\bar{T})\bar{T}(\theta^3\bar{T}) + 3(\theta^1\bar{T})^2\bar{T}^3 + 4(\theta^1\bar{T})^2\bar{T}(\theta^1\bar{T}) + (\theta^1\bar{T})^3\bar{T} \\
& + 3(\theta^1\bar{T})^2(\theta^2\bar{T}) + 6(\theta^1\bar{T})(\theta^2\bar{T})\bar{T}^2 + (\theta^1\bar{T})(\theta^2\bar{T})(\theta^1\bar{T}) \\
& + 3(\theta^1\bar{T})(\theta^3\bar{T})\bar{T} + (\theta^1\bar{T})(\theta^4\bar{T}) + (\theta^2\bar{T})\bar{T}^4 + 6(\theta^2\bar{T})\bar{T}^2(\theta^1\bar{T}) \\
& + 5(\theta^2\bar{T})\bar{T}(\theta^1\bar{T})\bar{T} + (\theta^2\bar{T})\bar{T}(\theta^2\bar{T}) + 4(\theta^2\bar{T})(\theta^1\bar{T})\bar{T}^2 \\
& + 3(\theta^2\bar{T})(\theta^1\bar{T})^2 + 3(\theta^2\bar{T})^2\bar{T} + 6(\theta^2\bar{T})(\theta^3\bar{T}) + (\theta^3\bar{T})\bar{T}^3 \\
& + 6(\theta^3\bar{T})\bar{T}(\theta^1\bar{T}) + 5(\theta^3\bar{T})(\theta^1\bar{T})\bar{T} + (\theta^3\bar{T})(\theta^2\bar{T}) + (\theta^4\bar{T})\bar{T}^2 \\
& + 6(\theta^4\bar{T})(\theta^1\bar{T}) + (\theta^5\bar{T})\bar{T} + (\theta^6\bar{T}) - f_{\theta^7}\bar{T}
\end{aligned}
\tag{3.4}
$$

4. On f_{θ^p} and p-rank in Genus 2

In this section, we give an explicit formula for f_{θ^p} on a genus 2 curve C, and note that we can use these ideas to derive explicit formulas for the p-rank of the Jacobian of C. Throughout, we work under the hypotheses and notation of Situations 2.2 and 3.1, with $X = C$.

We first note that (irrespective of the genus of C), although f_{θ^p} will be 0 only if $\theta(f) = 1$ for some f on U, we will always have:

Lemma 4.1. *$\theta f_{\theta^p} = 0$.*

Proof. Given any f, $\theta^p f = f_{\theta^p}\theta(f)$, so $\theta^{p+1}f = \theta(f_{\theta^p}\theta(f)) = \theta(f_{\theta^p})\theta(f) + f_{\theta^p}\theta^2(f) = \theta(f_{\theta^p})\theta(f) + \theta^{p+1}(f)$. Since this is true for all f, we must have $\theta(f_{\theta^p}) = 0$, as desired. □

We now specify some normalizations and notational conventions special to genus 2 which we will follow through the end of our explicit calculations in Section 7.

Situation 4.2. C is a smooth, proper genus 2 curve over an algebraically closed field k. It is presented explicitly on an affine open set U_2 by

$$y^2 = g(x) = x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5,$$

with the complement of U_2 being a single, smooth, Weierstrass point w at infinity. We also have the form $\omega_2 = y^{-1}dx$ trivializing Ω^1_C on U_2, and the derivation θ on U_2 given by $\theta f = y\frac{df}{dx}$. Equivalently, $\hat{\theta}(\omega_2) = 1$, where $\hat{\theta}$ denotes the map $\Omega^1_C \to \mathscr{O}_C$ such that $\theta = \hat{\theta} \circ d$.

For this section only, we set $U = U_2$ and $\omega = \omega_2$. We set $g_k(x) = \theta^{k-1}x$; we see by induction that this is a polynomial in x for k odd. Noting that $\theta(p(x)) = yp'(x)$ for $p(x)$ any polynomial in x, and $\theta(y) = \frac{1}{2}g'(x)$, we have that for k odd, $g_k(x) = \theta^2(g_{k-2}(x)) = \theta(yg'_{k-2}(x))$, and we get the recursive formula:

$$g_k(x) = g''_{k-2}(x)g(x) + \frac{1}{2}g'_{k-2}(x)g'(x) \tag{4.1}$$

for k odd.

But $f_{\theta^p} = \hat{\theta}^p(y^{-1}dx)$ by definition, which is just $y^{-1}\theta^p(x)$, so we also find

$$f_{\theta^p} = y^{-1}\theta g_p(x) = g'_p(x) \tag{4.2}$$

In particular, f_{θ^p} is a polynomial in x, and can therefore only have nonzero terms mod p in degrees which are multiples of p. However, we see by induction that the degree of $g_p(x)$ is always less than $2p$, so the only nonzero terms of f_{θ^p} are the constant term and the pth power term (from which it follows that the only nonzero terms of $g_p(x)$ are the constant, linear, pth power, and $(p+1)$st power terms).

For later use, we note the formulas for characteristics 3, 5, and 7 obtained by combining equations 4.1 and 4.2:

Characteristic 3:

$$f_{\theta^3} = x^3 + a_3 \tag{4.3}$$

Characteristic 5:

$$f_{\theta^5} = 2a_1x^5 + a_3^2 + 2a_2a_4 + 2a_1a_5 \tag{4.4}$$

Characteristic 7:

$$f_{\theta^7} = (3a_1^2 + 3a_2)x^7 + a_3^3 + 6a_2a_3a_4 + 3a_1a_4^2 + 3a2^2a_5 + 6a_1a_3a_5 + 6a_4a_5 \tag{4.5}$$

As a final note, we can use this to derive explicit formulas for the p-rank of the Jacobian of C in terms of the coefficients of $g(x)$.

Proposition 4.3. *If we denote by h_1, h_2, h_3, h_4 the polynomials in the coefficients of $g(x)$ giving the constant, linear, pth power, and $(p+1)$st power terms of $g_p(x)$, then the p-rank of the Jacobian of C is:*

2 if: $h_1h_4 - h_2h_3 \neq 0$;
1 if: $h_1h_4 - h_2h_3 = 0$ *but either* $h_3^p - h_2h_4^{p-1} \neq 0$ *or* $h_1^p h_4 - h_2^{p+1} \neq 0$;
0 if: $h_1h_4 - h_2h_3 = h_3^p - h_2h_4^{p-1} = h_1^p h_4 - h_2^{p+1} = 0$.

Proof. The p-torsion of the Jacobian is simply the number of (transport equivalence classes of, but endomorphisms of a line bundle are only scalars, and hold connections fixed) connections with p-curvature 0 on the trivial bundle. We note that the space of connections on $\mathscr{O}_C$ can be written explicitly as $f \mapsto df + f(c_1 + c_2x)\omega$, meaning the connection matrix on U with respect to θ is given simply by the function $\bar{T} = c_1 + c_2x$. Using the p-curvature formula for rank 1 given by Corollary 3.6 (i), we find

$$\psi_\nabla(\theta_0) = (c_1 + c_2x)^p + \theta_0^{p-1}(c_1 + c_2x) - f_{\theta_0^p}(c_1 + c_2x) \tag{4.6}$$

$$= c_1^p + c_2^p x^p + c_2 g_p(x) - g'_p(x)(c_1 + c_2x) \tag{4.7}$$

$$= (c_1^p + c_2h_1 - c_1h_2) + (c_2^p + c_2h_3 - c_1h_4)x^p. \tag{4.8}$$

Setting the p-curvature to zero, we obtain:

$$0 = (c_1^p + c_2h_1 - c_1h_2) + (c_2^p + c_2h_3 - c_1h_4)x^p.$$

We first consider this equation in the case that $h_4 \neq 0$. In this case, we find that we can write $c_1 = \frac{c_2^p + c_2 h_3}{h_4}$, and substituting in, we find we get p^2 solutions if $h_1 h_4^p - h_2 h_3 h_4^{p-1} \neq 0$, and otherwise, p solutions if $h_3^p - h_2 h_4^{p-1} \neq 0$, and finally 1 solution if both vanish. On the other hand, in the case that $h_4 = 0$, we see that c_2 becomes independent of c_1, we get p^2 solutions if and only if both h_2 and h_3 are nonzero; p solutions if either but not both are nonzero, and 1 solution if they are both 0. One can then check that both these casese are expressed by the asserted polynomial conditions in the h_i. □

For $p = 3$, we have

$$g_p(x) = 1x^4 - a_1 x^3 + a_3 x - a_4,$$

so h_4 is always nonzero, and we find that the p-rank of C is 2 when $a_4 - a_1 a_3 \neq 0$, is 1 when $a_4 - a_1 a_3 = 0$ but $a_1^3 - a_3 \neq 0$, and is 0 when $a_4 - a_1 a_3 = a_1^3 - a_3 = 0$.

For $p = 5$, we have

$$g_p(x) = 2a_1 x_6 + (4a_1^2 + 3a_2)x^5 + (a_3^2 + 2a_2 a_4 + 2a_1 a_5)x + (3a_3 a_4 + 3a_2 a_5),$$

so the p-rank of C is 2 when

$$a_1(a_3 a_4 + a_2 a_5) - (4a_1^2 + 3a_2)(a_3^2 + 2a_2 a_4 + 2a_1 a_5) \neq 0.$$

The p-rank is 1 when

$$a_1(a_3 a_4 + a_2 a_5) - (4a_1^2 + 3a_2)(a_3^2 + 2a_2 a_4 + 2a_1 a_5) = 0$$

but either

$$4a_1^{10} + 3a_2^5 - (a_3^2 + 2a_2 a_4 + 2a_1 a_5)a_1^4 \neq 0$$

or

$$(3a_3^5 a_4^5 + 3a_2^5 a_5^5)2a_1 - (a_3^2 + 2a_2 a_4 + 2a_1 a_5)^6 \neq 0.$$

Lastly, the p-rank is 0 when

$$\begin{aligned} 0 &= a_1(a_3 a_4 + a_2 a_5) - (4a_1^2 + 3a_2)(a_3^2 + 2a_2 a_4 + 2a_1 a_5) \\ &= 4a_1^{10} + 3a_2^5 - (a_3^2 + 2a_2 a_4 + 2a_1 a_5)a_1^4 \\ &= (3a_3^5 a_4^5 + 3a_2^5 a_5^5)2a_1 - (a_3^2 + 2a_2 a_4 + 2a_1 a_5)^6. \end{aligned}$$

While explicit computations of the p-rank of the Jacobian of a curve are not hard in general, it is perhaps worth mentioning that this method, aside from providing a complete and explicit solution for genus 2 curves, does so in a sufficiently elementary way that it can be presented as a calculation of the p-torsion of $\mathrm{Pic}(C)$ without knowing any properties of the Jacobian, or even that it exists.

5. The Space of Connections

In this section we carry out the first portion of the necessary computations for the explicit portion of Theorem 1.2, by calculating the space of transport-equivalence classes of connections on a particular vector bundle $\mathscr{E}$. We suppose:

Situation 5.1. With the notation and hypotheses of Situation 4.2, we further declare that $\mathscr{E}$ is the bundle determined by Propositions 2.3 and 2.6 for the choice $\mathscr{L} = \mathscr{O}_C([w])$.

In this situation, if U_1, U_2 are a trivializing cover for $\mathscr{L}$, with transition function φ_{12}, then $\mathscr{L}^{-1}, \mathscr{L}^{\otimes 2} = \Omega^1_C$, and $\mathscr{E}$ are all trivialized by this cover as well, and $\mathscr{E}$ can be represented with a transition matrix of the form

$$E = \begin{bmatrix} \varphi_{12} & \varphi_{\mathscr{E}} \\ 0 & \varphi_{12}^{-1} \end{bmatrix}$$

for some $\varphi_{\mathscr{E}}$ regular on $U_1 \cap U_2$.

We see immediately that we can choose φ_{12} and U_1 so that φ_{12} is regular on U_1 with a simple zero at w, and non-vanishing elsewhere: we simply set φ_{12} to be any function with a simple zero at w, and take U_1 to be the complement of any other zeroes and poles. For compatibility of trivializations of $\mathscr{L}$ and Ω^1_C, we must then set $\omega_1 = \varphi_{12}^{-2}\omega_2$. Beyond these properties, our specific choice of φ_{12} will be completely irrelevant, but we note that it is possible to choose φ_{12} to vary algebraically (in fact, to be in some sense invariant) as our a_i and the corresponding curves vary: we can simply set $\varphi_{12} = \frac{x^2}{y}$.

Proposition 5.2. *The unique non-trivial isomorphism class for $\mathscr{E}$ may be realized by setting $\varphi_{\mathscr{E}} = \varphi_{12}^{-2}$.*

Proof. We claim that there cannot be a splitting map from $\mathscr{E}$ back to $\mathscr{L}$. Indeed, one checks explicitly that such a splitting would require the existence of a rational function on C having a pole of order exactly 3 at w, and regular elsewhere, which is not possible. □

We now note that since φ_{12} has a simple zero at w, and ω_1 is invertible at w, if we further restrict U_1 we can guarantee that $\frac{d\varphi_{12}}{\omega_1}$ is likewise everywhere invertible on U_1. Having done so, $\varphi_{\mathscr{E}} = \varphi_{12}^{-2}$, so $d\varphi_{\mathscr{E}} = -2\varphi_{12}^{-3}d\varphi_{12}$, and $\frac{d\varphi_{\mathscr{E}}}{\omega_1}$ is regular and nonvanishing on U_1 except for a pole of order 3 at w.

Now, we can trivialize $\mathscr{E} \otimes \Omega^1_C$ on the U_i with transition matrix $\varphi_{12}^2 E$. We can then represent a connection $\nabla : \mathscr{E} \to \mathscr{E} \otimes \Omega^1_C$ by 2×2 connection matrices $\bar{T}_1$ and $\bar{T}_2$ of functions regular on U_1 and U_2 respectively. These act by sending $s_i \mapsto \bar{T}_i s_i + \frac{ds_i}{\omega_i}$ on U_i, where the s_i are given as vectors under the trivialization, so one checks that $\bar{T}_1$ and $\bar{T}_2$ must be related by:

$$\bar{T}_1 = \varphi_{12}^2 E \bar{T}_2 E^{-1} + E\frac{dE^{-1}}{\omega_1}$$

We now explicitly compute $\bar{T}_2$ in terms of $\bar{T}_1$ in preparation for computing the space of connections. If $\bar{T}_2 = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$, then:

$$\bar{T}_1 = \begin{bmatrix} \varphi_{12}^2 f_{11} + f_T & \varphi_{12}^4 f_{12} + \varphi_{12}^3 \varphi_{\mathscr{E}}(f_{22} - f_{11}) - \varphi_{12}\varphi_{\mathscr{E}} f_T - \varphi_{12}\frac{d\varphi_{\mathscr{E}}}{\omega_1} \\ f_{21} & \varphi_{12}^2 f_{22} - f_T \end{bmatrix} \tag{5.1}$$

where $f_T = \varphi_{12}\varphi_{\mathscr{E}} f_{21} - \varphi_{12}^{-1}\frac{d\varphi_{12}}{\omega_1}$

Note that this implies f_{21} is everywhere regular and hence constant.

We now show:

Proposition 5.3. *The space of connections on $\mathscr{E}$ is given by $f_{21} = C_1$, $f_{11} = c_1 + c_2 x$, $f_{22} = c_3 + c_4 x$, and $f_{12} = c_5 + c_6 x + c_7 x^2 + c_8 y + C_2 x^3$, where the c_i are arbitrary constants subject to the single linear relation $c_8 = C_2(c_2 - c_4)$, and C_1 and C_2 are predetermined nonzero constants satisfying $C_1 C_2 = \frac{-1}{2}$.*

Proof. We begin by looking at the lower right entry of the matrix for $\bar{T}_1$ in Equation 5.1, and note the $\varphi_{12}^{-1}\frac{d\varphi_{12}}{\omega_1}$ has a simple pole at w which must be cancelled by one of the other terms. We also note that since $\varphi_{\mathscr{E}} = \varphi_{12}^{-2}$, and f_{21} must be constant, the term $\varphi_{12}\varphi_{\mathscr{E}}f_{21} = \varphi_{12}^{-1}f_{21}$ is regular on U_1 away from w, where it can have at most a simple pole. Thus the $\varphi_{12}^2 f_{22}$ term must likewise be regular on U_1 away from w, with at most a simple pole at w. Since f_{22} must be regular on U_2 by hypothesis, we conclude it is regular on C except possibly for a pole of order at most 3 at w. But such a pole of order 3 isn't possible, so $f_{22} \in \Gamma(\mathscr{O}_C(2[w]))$. This means that the simple poles of the other two terms must cancel, and f_{21} is determined as a (nonzero) constant C_1: explicitly, $C_1 = \frac{d\varphi_{12}}{\omega_1}(w)$. Precisely the same argument applies to the upper right entry, placing $f_{11} \in \Gamma(\mathscr{O}_C(2[w]))$, so it only remains to analyze the upper right entry of the matrix.

We immediately observe that on U_1, each term (excluding the $\varphi_{12}^4 f_{12}$ term) is regular except possibly for a pole of order at most 2 at w, which of course implies that $\varphi_{12}^4 f_{12}$ is also, and we can conclude that f_{12} is regular on C except for a pole of order at most 6 at w. Then we have $f_{21} = C_1 \in k^*$, $f_{11} = c_1 + c_2 x$, $f_{22} = c_3 + c_4 x$, and $f_{12} = c_5 + c_6 x + c_7 x^2 + c_8 y + C_2 x^3$, and we claim that C_2 is also determined: the only other terms which can have double poles are $-\varphi_{12}^2\varphi_{\mathscr{E}}^2 f_{21} + \varphi_{\mathscr{E}}\frac{d\varphi_{12}}{\omega_1} - \varphi_{12}\frac{d\varphi_{\mathscr{E}}}{\omega_1} = -\varphi_{12}^{-2}f_{21} + 3\varphi_{12}^{-2}\frac{d\varphi_{12}}{\omega_1}$ which are now predetermined, so C_2 is also determined, explicitly as $-2(\varphi_{12}^{-6}x^{-3})(w)C_1$. Lastly, we note that there is a linear relation on c_2, c_4, and c_8 to insure that the simple poles cancel.

To conclude the proof, we use formal local analysis at w to obtain the desired statements on this linear relation and C_1 and C_2. Explicitly, our linear relation is given as $c_8 = ((\varphi_{12}^{-3}y^{-1}x)(w))(c_2 - c_4) + ((y^{-1}\varphi_{12}^{-5})(w))((\varphi_{12}^{-1}(C_1 - 3\frac{d\varphi_{12}}{\omega_1}) - C_2 x^3\varphi_{12}^5)(w))$. Now, choose a local coordinate z at w; we will denote by $\ell_z(f)$ and $\ell'_z(f)$ the leading and second terms of the Laurent series expansion for f in terms of z. From our relation between x and y, we have $\ell_z(x)^5 = \ell_z(y)^2$ and $2\ell_z(y)\ell'_z(y) = 5\ell_z(x)^4\ell'_z(x)$. Simply considering leading terms, we find that since $\omega_1 = \varphi_{12}^{-2}y^{-1}dx$, we have $C_1 = \frac{-\ell_z(\varphi_{12})^3\ell_z(y)}{2\ell_z(x)}$, and $C_2 = \frac{\ell_z(x)}{\ell_z(\varphi_{12})^3\ell_z(y)}$. Thus, we have that $C_1C_2 = \frac{-1}{2}$, and also that C_2 is the coefficient of $(c_2 - c_4)$ in our linear relation. It only remains to show that the constant term in that relation is in fact 0. We may write it as $((y^{-1}\varphi_{12}^{-5})(w))((\varphi_{12}^{-1}(C_1 - 3\frac{d\varphi_{12}}{\omega_1} - C_2x^3\varphi_{12}^6))(w)$, so it suffices to show that $C_1 - 3\frac{d\varphi_{12}}{\omega_1} - C_2x^3\varphi_{12}^6$, which we know must vanish at w, in fact vanishes to order at least 2 at w. For this, it is convenient to specialize to $z = \varphi_{12}$, whereupon our earlier relation simplifies to $\ell_z(y) = C_2^{-1}\ell_z(x)$. We also compute $\ell_z(x)^3 = C_2^{-2}$, from which it follows that we can write $\ell'_z(y) = \frac{5}{2}C_2^{-1}\ell'_z(x)$. We can now write everything in terms of $\ell_z(x), \ell'_z(x)$ and C_2, and check directly that we get the desired cancellation to order 2. $\square$

We also consider the endomorphisms of $\mathscr{E}$, so that we can normalize our connections via transport to simplify calculations. An endomorphism is given by matrices S_i regular on U_i, satisfying the relationship $S_1 = ES_2E^{-1}$. If we write $S_2 = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$ we find that

$$S_1 = \begin{bmatrix} g_{11} + \varphi_{12}^{-1}\varphi_{\mathscr{E}}g_{21} & \varphi_{12}^2 g_{12} + \varphi_{12}\varphi_{\mathscr{E}}g_{22} - \varphi_{12}\varphi_{\mathscr{E}}g_{11} - \varphi_{\mathscr{E}}^2 g_{21} \\ \varphi_{12}^{-2}g_{21} & g_{22} - \varphi_{12}^{-1}\varphi_{\mathscr{E}}g_{21} \end{bmatrix} \tag{5.2}$$

We can now compute directly:

Proposition 5.4. *The space of endomorphisms of $\mathscr{E}$ is given by $g_{21} = 0$, $g_{11} = g_{22} \in k$, and $g_{12} = g_{12}^0 + g_{12}^1 x \in \Gamma(\mathscr{O}_C(2[w]))$. Every connection on $\mathscr{E}$ has a unique transport-equivalent connection with $f_{11} = 0$.*

Proof. Noting that the lower left entry for S_1 in equation 5.2 is $\varphi_{12}^{-2} g_{21}$, we see that g_{21} has to be regular everywhere on C, and vanishes to order at least 2 at w; hence, it is 0. We then see that the upper left and lower right entries are just g_{11} and g_{22} respectively, meaning that these are both everywhere regular and hence constant. Finally, the upper right term is then $\varphi_{12}^2 g_{12} + \varphi_{12}^{-1}(g_{22} - g_{11})$; the second term will have a simple pole at w if and only if $g_{22} \neq g_{11}$, and since g_{12} cannot have a triple pole at w, we conclude that $g_{22} = g_{11}$, and finally that $g_{12} \in \Gamma(\mathscr{O}_C(2[w]))$, giving the description of the endomorphisms of $\mathscr{E}$.

Such an endomorphism is invertible if and only if $g_{11} \neq 0$. Since transport along an automorphism is invariant under scaling the automorphism, we can then set $g_{11} = g_{22} = 1$ without loss of generality. Now, since S_2 is upper triangular, with constant diagonal coefficients, $S_2^{-1} \frac{dS_2}{\omega_2}$ has only its upper right coefficient non-zero. Moreover, conjugating $\bar{T}_2$ by S_2 will simply substract $f_{21} g_{12}$ from the upper left coefficient of $\bar{T}_2$. Since we know f_{21} is a determined nonzero constant, and g_{12} and f_{11} can both be arbitrary in $\Gamma(\mathscr{O}_C(2[w]))$, this means that each connection has a unique transport class with $f_{11} = 0$, as desired. □

Thus, from now on we will normalize our calculations as follows: set $f_{11} = 0$ by transport; set $f_{22} = 0$ since we want the determinant connection (obtained by taking the trace) to be 0; and set $f_{21} = 1$. We accomplish the last by scaling φ_{12} appropriately: we saw that $f_{21} = \frac{d\varphi_{12}}{\omega_1}(w)$, and recalling that $\omega_1 = \varphi_{12}^{-2} y^{-1} dx$, it suffices to scale φ_{12} by a cube root of f_{21}. We also note that this does not pose any problems for our prior choice of $\varphi_{12} = \frac{x^2}{y}$; one can check that for this choice, we have f_{21} invariant as $\frac{-1}{2}$, and the scaling factor for φ_{12} is independent of the a_i. Lastly, since $c_8 = 0$ now that $c_2 = c_4 = 0$, we conclude that we are reduced to considering the case:

Situation 5.5. Our connection matrix T_2 on U_2 is of the form $T_2 = \begin{bmatrix} 0 & f_{12} \\ 1 & 0 \end{bmatrix}$, with $f_{12} = c_5 + c_6 x + c_7 x^2 - \frac{1}{2} x^3$.

Finally, for later use we formally generalize our results.

Proposition 5.6. *Propositions 5.3 and 5.4 hold in the following more more general settings:*

(i) *After base change to an arbitrary k-algebra A, if we replace the k-valued constants by A-valued constants;*

(ii) *When we allow our defining polynomial $g(x)$ to degenerate to produce nodes away from w, if we replace Ω^1_C by the dualizing sheaf ω_C in the definition of connections;*

(iii) *When we consider families of curves obtained from maps $k[a_1, \dots a_5] \to A$ taking values in the open subset $U_{\mathrm{nod}} \subset \mathbb{A}^5$ corresponding to at worst nodal curves.*

Proof. For (i), if we denote by f the map $\operatorname{Spec} A \to \operatorname{Spec} k$, and π the structure map $C \to \operatorname{Spec} k$, this coefficient replacement corresponds to the natural map $f^* \pi_* \mathscr{F} \to \pi_{f*} f_\pi^* \mathscr{F}$ for the sheaves $\mathscr{E}nd(\mathscr{E}) \otimes \Omega^1_C$ and $\mathscr{E}nd(\mathscr{E})$. But since the base

is a point, every base change is flat, and it immediately follows [3, Prop. III.9.3] that this natural map is always an isomorphism, giving the desired statement.

For (ii), we need only note that our arguments go through unmodified, since ω_C is still isomorphic to $\mathscr{O}(2[w])$, and the same standard Riemann-Roch argument as in the smooth case still shows it that there can be no function in $\Gamma(\mathscr{O}(3[w])) \smallsetminus \Gamma(\mathscr{O}(2[w]))$.

Finally, for (iii) we make use of the fact that, as remarked immediately above, we can choose φ_{12} to be a specific function varying algebraically in the whole family. Once again, if we denote by $\mathscr{F}$ the sheaf $\mathcal{E}nd(\mathscr{E}) \otimes \omega_C$ or $\mathcal{E}nd(\mathscr{E})$ as appropriate, but this time in the universal setting over U_{nod}, the theory of cohomology and base change gives that since $h^0(C, \mathscr{F})$ is constant on fibers, $\pi_*\mathscr{F}$ is locally free of the same rank, and pushforward commutes with base change. Now, if we let our constants describing sections of $\mathscr{F}$ lie in $k[a_1, \ldots, a_5]$, we clearly obtain a subsheaf of $\pi_*\mathscr{F}$ of the correct rank; further, the inclusion map is an isomorphism when restricted to every fiber, so it must in fact be an isomorphism, which yields the desired result for arbitrary A via base change. □

It follows formally that the closed subschemes we describe explicitly corresponding to vanishing p-curvature in Section 6 and nilpotent p-curvature in Section 7 are also functorial descriptions which hold for nodal curves.

6. Calculations of p-curvature

Continuing with the situation and notations of the previous section, and in particular that of Situation 4.2, we conclude with the p-curvature calculations to complete the proof of Theorem 1.2 for $p \leq 7$, except for the statement on the general curve in characteristic 7, which depends on the results of the subsequent section.

We write:

$$\psi_\nabla(\theta) = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$$

The first case we handle is $p = 3$. Equation 4.3 gave us $f_{\theta^3} = x^3 + a_3$. We show:

Proposition 6.1. *In characteristic* 3, *$\mathscr{E}$ has a unique transport equivalence class of connections with p-curvature zero and trivial determinant.*

Proof. With all of our normalizations from Situation 5.5, the p-curvature matrix given by Equation 3.2 becomes rather tame:

$$\psi_\nabla(\theta) = \begin{bmatrix} \theta f_{12} & f_{12}^2 + \theta^2 f_{12} - f_{\theta^3} f_{12} \\ f_{12} - f_{\theta^3} & -\theta f_{12} \end{bmatrix}$$

Even better, we note that we have

$$h_{12} = \theta(h_{11}) + f_{12} h_{21},$$

so h_{12} vanishes if h_{11} and h_{21} do. Similarly, recalling that by Lemma 4.1, $\theta f_{\theta^3} = 0$, we see that $h_{11} = \theta(h_{21})$, and $h_{22} = -h_{11}$. Hence, to check if the p-curvature vanishes, it suffices to check that h_{21} vanishes.

But this is a triviality, as we simply get that $h_{21} = 0$ if and only if $f_{12} = a_3 + x^3$. Recalling that after normalization f_{12} was given by $c_5 + c_6 x + c_7 x^2 - \frac{1}{2} x^3$, we get the unique solution $c_5 = a_3, c_6 = c_7 = 0$. □

We now handle the case $p = 5$. We had from Equation 4.4 that $f_{\theta^5} = 2a_1 x^5 + a_3^2 + 2a_2 a_4 + 2a_1 a_5$.

Proposition 6.2. *In characteristic* 5*, the number of transport equivalence classes of connections with* p*-curvature zero and trivial determinant is given as the number of roots of the quintic polynomial:*

$$(3a_1a_2^2 + 3a_2a_3 + a_5) + (a_1^2a_2 + a_2^2 + 3a_1a_3 + 4a_4)c_5 \\ + (3a_1^3 + 4a_1a_2 + a_3)c_5^2 + (3a_1^2 + 4a_2)c_5^3 + a_1c_5^4 + 4c_5^5$$

Proof. With our normalizations as above, in terms of f_{12} and f_{θ^5}, the p-curvature matrix obtained from Equation 3.3 is

$$\psi_\nabla(\theta) = \begin{bmatrix} 4f_{12}\theta(f_{12}) + \theta^3(f_{12}) & f_{12}^3 + 4(\theta(f_{12}))^2 + 2f_{12}\theta^2(f_{12}) + \theta^4(f_{12}) + 4f_{12}f_{\theta^5} \\ f_{12}^2 + 3\theta^2(f_{12}) + 4f_{\theta^5} & f_{12}\theta(f_{12}) + 4\theta^3(f_{12}) \end{bmatrix}$$

Conveniently, we note that as before it actually suffices to check that h_{21} is 0, since we see that $h_{22} = 3\theta_0(h_{21})$, that $h_{11} = -h_{22}$, and that $h_{12} = f_{12}h_{21} + 2\theta^2(h_{21})$.

Substituting in for f_{12} and f_{θ^5}, we get that the remaining (lower left) term is given by

$$(4a_3^2 + 3a_2a_4 + 3a_1a_5 + c_3^2 + a_5c_5) + (a_5 + 3a_3c_4 + 2c_3c_4 + 4a_4c_5)x \\ + (2a_2c_4 + c_4^2 + 2a_3c_5 + 2c_3c_5)x^2 \\ + (4a_3 + 4c_3 + a_1c_4 + 2c_4c_5)x^3 + (3a_2 + 4c_4 + 3a_1c_5 + c_5^2)x^4$$

Setting the x^3 and x^4 terms to 0 allows us to solve for c_4 and c_3. Substituting in, we find that the x^2 term drops out, while the coefficient of x is:

$$(3a_1a_2^2 + 3a_2a_3 + a_5) + (a_1^2a_2 + a_2^2 + 3a_1a_3 + 4a_4)c_5 \\ + (3a_1^3 + 4a_1a_2 + a_3)c_5^2 + (3a_1^2 + 4a_2)c_5^3 + a_1c_5^4 + 4c_5^5$$

The constant coefficient is $c_5 + 3a_1$ times the x coefficient, so we get that the connections with p-curvature 0 correspond precisely to the roots of the above polynomial, as asserted. □

Lastly, we take a look at the case $p = 7$. Equation 4.5 gave us:

$$f_{\theta^7} = a_3^3 + 6a_2a_3a_4 + 3a_1a_4^2 + 3a_2^2a_5 + 6a_1a_3a_5 + 6a_4a_5 + (3a_1^2 + 3a_2)x^7.$$

We will show:

Proposition 6.3. *In characteristic 7, the number of transport equivalence classes of connections on* $\mathscr{E}$ *with* p*-curvature* 0 *and trivial determinant is given as the intersection of four plane curves in* $\mathbb{A}^2$*. For a general curve, it is positive. The locus* $F_{2,7}$ *of transport equivalence classes of connections on* $\mathscr{E}$ *with* p*-curvature* 0 *and trivial determinant considered over the* $\mathbb{A}^5$ *with which we parametrize genus* 2 *curves is cut out by* 4 *hypersurfaces in* $\mathbb{A}^5 \times \mathbb{A}^2$*.*

Proof. Here, even with our normalizations the p-curvature matrix obtained from Equation 3.4 is rather messy, but we find its coefficients are given by:

$$h_{11} = 2f_{12}^2\theta(f_{12}) + \theta(f_{12})\theta^2(f_{12}) - 3f_{12}\theta^3(f_{12}) + \theta^5(f_{12})$$

$$h_{21} = -f_{\theta^7} + f_{12}^3 + 3(\theta(f_{12}))^2 - f_{12}\theta^2(f_{12}) - 2\theta^4(f_{12})$$

$$h_{12} = -f_{\theta^7} f_{12} + f_{12}^4 + f_{12}^2\theta^2(f_{12}) + (\theta^2(f_{12}))^2 - 2\theta(f_{12})\theta^3(f_{12}) + 2f_{12}\theta^4(f_{12}) + \theta^6(f_{12})$$

$$h_{22} = -2f_{12}^2\theta(f_{12}) - \theta(f_{12})\theta^2(f_{12}) + 3f_{12}\theta^3(f_{12}) - \theta^5 f_{12}$$

Once again, it is enough to consider a single one of these coefficients, as we see that $h_{11} = 3\theta(h_{21})$, that $h_{12} = f_{12}h_{21} + 3\theta^2(h_{21})$, and that $h_{22} = -h_{11}$.

Looking then at the formula for h_{21}, substituting in for f_{12} and f_{θ^7} gives a polynomial of degree 6 in x. The x^6 term lets us solve for c_3:

$$c_3 = 5a_1a_2 + a_3 + 4a_1c_4 + 4a_1^2c_5 + c_4c_5 + 2a_1c_5^2 + 5c_5^3$$

The x^5 term is then

$$\begin{aligned} h_{7,1} = {} & 2a_1^2a_2 + a_1a_3 + 5a_4 + 4a_1^2c_4 + 5a_2c_4 + 6c_4^2 \\ & + 3a_1^3c_5 + 6a_1a_2c_5 + 3a_3c_5 + 5a_1c_4c_5 + 3a_1c_5^3 + 6c_5^4. \end{aligned} \tag{6.1}$$

while the x^4 term is $-c_5$ times the x^5 term, and the x^3 term is $-(c_5^2+a_1c_5+3a_2+c_4)$ times the x^5 term. Taking the x^2 term minus $-(5c_5^3+5a_1c_5^2+2c_4c_5+5a_1a_2+4a_3+2a_1c_4)$ times the x^5 term leaves:

$$\begin{aligned} h_{7,2} = {} & 3a_1^3a_2^2 + 6a_1^2a_2a_3 + 4a_1a_3^2 + 4a_3a_4 + 2a_2a_5 + 3a_1^3a_2c_4 + 4a_1^2a_3c_4 \\ & + 2a_1a_4c_4 + 4a_5c_4 + a_1^3c_4^2 + a_3c_4^2 + 3a_1c_4^3 + a_1^4a_2c_5 \\ & + 5a_1^3a_3c_5 + a_1^2a_4c_5 + 3a_1a_5c_5 + 6a_1^4c_4c_5 + a_1^2a_2c_4c_5 \\ & + a_1a_3c_4c_5 + 3a_4c_4c_5 + a_1^2c_4^2c_5 + 5a_2c_4^2c_5 + c_4^3c_5 \\ & + 4a_1^3a_2c_5^2 + 6a_1^2a_3c_5^2 + a_1a_4c_5^2 + 3a_5c_5^2 + 3a_1^3c_4c_5^2 \\ & + a_1a_2c_4c_5^2 + a_3c_4c_5^2 + 3a_1^2a_2c_5^3 + a_1a_3c_5^3 + 4a_1^2c_4c_5^3 + 6c_4^2c_5^3. \end{aligned} \tag{6.2}$$

Similarly, taking the x term minus $-(5c_4c_5^2 + 5a_1c_4c_5 + 6a_4 + 2c_4^2)$ times the x^5 term leaves:

$$\begin{aligned} h_{7,3} = {} & 5a_1^2a_2a_4 + 6a_1a_3a_4 + a_1a_2a_5 + 5a_1^2a_2^2c_4 + 4a_1a_2a_3c_4 + 3a_1^2a_4c_4 \\ & + 2a_1a_5c_4 + 5a_1^2a_2c_4^2 + a_1a_3c_4^2 + 3a_4c_4^2 + 3a_2c_4^3 \\ & + 5c_4^4 + 4a_1^3a_4c_5 + 6a_3a_4c_5 + 5a_1^2a_5c_5 + 3a_2a_5c_5 \\ & + 4a_1^3a_2c_4c_5 + 4a_1^2a_3c_4c_5 + a_1a_4c_4c_5 + 6a_5c_4c_5 \\ & + 3a_1^3c_4^2c_5 + 4a_1a_2c_4^2c_5 + 4a_3c_4^2c_5 + a_1c_4^3c_5 \\ & + 2a_1^2a_4c_5^2 + 6a_1a_5c_5^2 + 2a_1^2a_2c_4c_5^2 + 2a_1a_3c_4c_5^2 \\ & + a_4c_4c_5^2 + 5a_1^2c_4^2c_5^2 + 4a_2c_4^2c_5^2 + 5c_4^3c_5^2 \\ & + 5a_1a_4c_5^3 + a_5c_5^3 + 5a_1a_2c_4c_5^3 + 5a_3c_4c_5^3 + 2a_1c_4^2c_5^3. \end{aligned} \tag{6.3}$$

Lastly, taking the constant term minus

$$\begin{aligned}-(&6c_5^5+5c_4c_5^3+3a_1^2c_5^3+2a_1^3c_5^2+5a_1a_2c_5^2+2a_3c_5^2+6a_1c_4c_5^2\\&+5a_1^2a_2c_5+2a_1a_3c_5+2a_4c_5+a_1^2c_4c_5\\&+2a_2c_4c_5+2c_4^2c_5+6a_1a_4+4a_5+4a_1a_2c_4+3a_3c_4+3a_1c_4^2)\end{aligned}$$

times the x^5 term leaves:

$$\begin{aligned}h_{7,4}=\;&6a_1^3a_2^3+5a_1^2a_2^2a_3+a_1a_2a_3^2+5a_1^3a_2a_4+6a_1^2a_3a_4+a_2a_3a_4\\&+6a_1a_4^2+a_1^2a_2a_5+4a_2^2a_5+5a_1a_3a_5+4a_4a_5\\&+4a_1^2a_2a_3c_4+a_1a_3^2c_4+3a_1^3a_4c_4+2a_1a_2a_4c_4\\&+3a_3a_4c_4+2a_1^2a_5c_4+3a_1^3a_2c_4^2+6a_1a_2^2c_4^2\\&+a_2a_3c_4^2+6a_5c_4^2+6a_1^3c_4^3+4a_1a_2c_4^3+4a_3c_4^3\\&+4a_1c_4^4+2a_1^4a_2^2c_5+3a_1^3a_2a_3c_5+4a_1^4a_4c_5\\&+2a_1^2a_2a_4c_5+2a_1a_3a_4c_5+5a_1^3a_5c_5+a_3a_5c_5+3a_1^4a_2c_4c_5\\&+2a_1^2a_2^2c_4c_5+2a_1^3a_3c_4c_5+2a_1a_2a_3c_4c_5+5a_3^2c_4c_5\\&+6a_1^2a_4c_4c_5+6a_2a_4c_4c_5+5a_1a_5c_4c_5+2a_1^4c_4^2c_5+2a_1^2a_2c_4^2c_5\\&+3a_2^2c_4^2c_5+6a_1a_3c_4^2c_5+4a_4c_4^2c_5+a_2c_4^3c_5+5c_4^4c_5\\&+a_1^3a_2^2c_5^2+5a_1^2a_2a_3c_5^2+2a_1^3a_4c_5^2+2a_1a_2a_4c_5^2\\&+2a_3a_4c_5^2+6a_1^2a_5c_5^2+5a_1^3a_2c_4c_5^2+2a_1a_2^2c_4c_5^2\\&+a_1^2a_3c_4c_5^2+2a_2a_3c_4c_5^2+4a_1a_4c_4c_5^2+5a_5c_4c_5^2\\&+a_1^3c_4^2c_5^2+6a_1a_2c_4^2c_5^2+2a_1c_4^3c_5^2+6a_1^2a_2^2c_5^3\\&+2a_1a_2a_3c_5^3+5a_1^2a_4c_5^3+a_1a_5c_5^3+2a_1^2a_2c_4c_5^3\\&+6a_1a_3c_4c_5^3+5a_4c_4c_5^3+6a_1^2c_4^2c_5^3+4a_2c_4^2c_5^3+3c_4^3c_5^3.\end{aligned}\tag{6.4}$$

These four polynomials are then the defining equations in characteristic 7, describing the locus as an intersection of 4 affine plane curves, as desired. By direct computation in Macaulay 2, the coordinate ring of the affine algebraic set cut out by these equations has dimension 5. Since we know that it can only have dimension 0 over any given choice for the a_i, this implies that it has a non-empty fiber for a general choice of a_i, yielding the positivity assertion. □

Finally, we compute an example which will allow us to deduce the characteristic 7 case of Theorem 1.2 in the next section.

Lemma 6.4. *For the curve given by $a_1=a_2=a_3=0, a_4=1$, and $a_5=3$, there are 14 solutions to our equations, all reduced. Further, the local rings of $F_{2,7}$ at each of these points are all isomorphic.*

Proof. First, we set $a_1=a_2=a_3=0$, $a_4=1$ and $a_5=3$, and our defining equations become considerably simpler:

$$\begin{aligned}h_{7,1}&=5+6c_4^2+6c_5^4\\h_{7,2}&=5c_4+3c_4c_5+c_4^3c_5+2c_5^2+6c_4^2c_5^3\end{aligned}$$

$$h_{7,3} = 3c_4^2 + 5c_4^4 + 4c_4c_5 + c_4c_5^2 + 5c_4^3c_5^2 + 3c_5^3$$
$$h_{7,4} = 5 + 4c_4^2 + 4c_4^2c_5 + 5c_4^4c_5 + c_4c_5^2 + 5c_4c_5^3 + 3c_4^3c_5^3$$

If we use $h_{7,1}$ to substitute for c_4^2 in $h_{7,2}$, we get:

$$c_4(5 + c_5 + 6c_5^5) + c_5^2(2 + 2c_5 + c_5^5)$$

We check that we cannot have $5+c_5+6c_5^5 = 0$, so we can localize away from $5+c_5+6c_5^5$, setting $c_4 = \frac{c_5^2(2+2c_5+c_5^5)}{5+c_5+6c_5^5}$. Making this substitution and taking numerators, the $h_{7,i}$ give four polynomials in c_5. However, they are multiples of the polynomial given by $h_{7,1}$, which is:

$$6 + c_5 + 5c_5^2 + 6c_5^4 + 2c_5^5 + 6c_5^6 + 6c_5^9 + 3c_5^{10} + 5c_5^{14}$$

This then gives the 14 reduced solutions, and the fact that the local rings of $F_{2,7}$ at each of these points are isomorphic follows from the fact that this degree 14 polynomial is irreducible over $\mathbb{F}_7$, since the 14 points are then Galois conjugate in $F_{2,7}$, which is defined over $\mathbb{F}_7$. □

7. On The Determinant of the p-Curvature Map

In this section we explicitly calculate the highest degree terms of $\det\psi$, the determinant of the p-curvature map in the case of a genus 2 curve and the specific unstable vector bundle of Situation 5.1. We use the calculation to prove that $\det\psi$ is finite flat, of degree p^3, and therefore conclude that in families of curves, the kernel of $\det\psi$ is finite flat. This has immediate implications for the connections on $\mathscr{E}$ of vanishing p-curvature as well, in particular allowing us to finish the proof of the characteristic-specific portion of Theorem 1.2. The results here are a special case of Mochizuki's work (see [10, Thm. II.2.3, p. 129]), obtained by an argument which is essentially the same, but discovered independently, and significantly simpler in the special case handled here.

We wish to compute in our specific situation the morphism $\det\psi^0$ (Proposition 2.8 (iv)), which is to say, the morphism obtained from ψ^0 (Proposition 2.8 (iii)) by taking the determinant. In fact, we take ψ^0 to be the induced map on transport-equivalence classes of connections. We remark that in the situation of rank 2 vector bundles with trivial determinant, and after restricting to connections with trivial determinant, because the image of ψ^0 is contained among the traceless endomorphisms, the vanishing of the determinant of the p-curvature is then equivalent to nilpotence of the endomorphisms given by the p-curvature map. Such connections are frequently called **nilpotent** in the literature (see, for instance, [7] or [11]).

We now take our curve C of genus 2 from before, with $\mathscr{E}$ the particular unstable bundle of rank 2 we had been studying, as in Situations 4.2, 5.1, and 5.5. We also take the particular θ from before, with $\hat{\theta}(\omega_2) = 1$. Since ω_2 has a double zero at w, we see that θ has a double pole there, so that our explicit identification of Ω^1_C is as $\mathscr{O}(2[w])$. We know that our space of connections with trivial determinant on $\mathscr{E}$ is (modulo transport) 3-dimensional, and of course $h^0(C^{(p)}, (\Omega^1_{C^{(p)}})^{\otimes 2}) = \deg(\Omega^1_{C^{(p)}})^{\otimes 2} + 1 - g = 4g - 4 + 1 - g = 3g - 3 = 3$, so we have a map from $\mathbb{A}^3$ to $\mathbb{A}^3$. We choose coordinates on the first space to be given by the (c_5, c_6, c_7) determining f_{12}, while the function we will get will be of the form $f_1(c_5, c_6, c_7) + f_2(c_5, c_6, c_7)x^p + f_3(c_5, c_6, c_7)x^{2p}$, and we obtain coordinates on the image space as the monomials $(1, x^p, x^{2p})$.

We will use our earlier calculations to recover, in a completely explicit and elementary fashion, the genus 2 case of Mochizuki's result:

Theorem 7.1. *On the unstable vector bundle $\mathscr{E}$ described by Situation 5.1 for a smooth proper genus 2 curve C as in Situation 4.2, the map $\det\psi^0$ is a finite flat morphism from $\mathbb{A}^3$ to $\mathbb{A}^3$, of degree p^3. Further, $\det\psi^0$ remains finite flat when considered as a family of maps over the open subset $U_{\mathrm{ns}} \subset \mathbb{A}^5$ corresponding to nonsingular curves. Lastly, the induced map from $\ker\det\psi^0$ to U_{ns} is finite flat.*

Proof. It suffices to prove the asserted finite flatness for the family of maps $\mathbb{A}^3 \times U_{\mathrm{ns}} \to \mathbb{A}^3 \times U_{\mathrm{ns}}$ over U_{ns}, since the statements on individual curves and on the kernel of $\det\psi$ both follow from restriction to fibers. This is turn will follow from the claim that the leading term of f_i is $-c_{i+4}^p$, with all other terms of strictly lesser total degree in the c_i. We prove this by direct calculation.

If $T = \begin{bmatrix} 0 & f_{12} \\ 1 & 0 \end{bmatrix}$ is the connection matrix for ∇, we claim that the leading term will come from the T^p term in the p-curvature formula. Now, $T^2 = \begin{bmatrix} f_{12} & 0 \\ 0 & f_{12} \end{bmatrix}$, so we find

$$T^p = \begin{bmatrix} 0 & (f_{12})^{\frac{p+1}{2}} \\ (f_{12})^{\frac{p-1}{2}} & 0 \end{bmatrix}$$

Next, f_{12} is linear in the c_i, as are $\theta^i f_{12}$ for all i. Considering the p-curvature formula coefficients as polynomials in $\theta^i f_{12}$, we will show that the degree of the remaining terms are all less than or equal to $\frac{p-1}{2}$, with the degree of the terms in the lower left strictly less. This will imply that the leading term of the determinant is given by

$$-(f_{12})^p = -(c_5 + c_6 x + c_7 x^2 - \frac{1}{2}x^3)^p = -c_5^p - c_6^p x^p - c_7^p x^{2p} + \frac{1}{2^p}x^{3p}$$

giving the desired formula for the leading terms of the constant, x^p, and x^{2p} terms.

We observe that since $\theta^i T = \begin{bmatrix} 0 & \theta^i f_{12} \\ 0 & 0 \end{bmatrix}$ for all $i > 0$, $(\theta^i T)(\theta^j T) = 0$ for any $i, j > 0$. We use this and the fact that T^2 is diagonal to write any term in the p-curvature as one of the following:

(1) $T^{2i_0}(\theta^{i_1}T)T\dots(\theta^{i_k}T)$
(2) $T^{2i_0}T(\theta^{i_1}T)T\dots(\theta^{i_k}T)$
(3) $T^{2i_0}(\theta^{i_1}T)T\dots(\theta^{i_k}T)T$
(4) $T^{2i_0}T(\theta^{i_1}T)T\dots(\theta^{i_k}T)T$

where $2i_0 + \sum_{j>0}(i_j + 2) = p+1, p, p, p-1$ respectively.

We observe that these correspond to non-zero upper right, lower right, upper left, and lower left coefficients, respectively (in particular, at most one is non-zero). We know that the first term is a scalar matrix of degree i_0 in f_{12}. We see that $T(\theta^{i_j}T) = \begin{bmatrix} 0 & 0 \\ 0 & \theta^i f_{12} \end{bmatrix}$, so a product of $k-1$ such terms has total degree $k-1$ in the $\theta^i f_{12}$. Lastly, multiplying on the left by $(\theta^{i_1}T)$ raises the degree by one and moves the nonzero coefficient back to the upper right. Thus, in the first case, we get total degree $i_0 + k$. But we see that this is actually the same in the other cases, as multiplying on the left or right by T just moves the nonzero coefficient, without changing it. Finally, with $k > 0$, we have $i_0 + k < \frac{1}{2}(2i_0 + \sum_{j>0}(i_j + 2))$, which

is $\frac{1}{2}$ times $p+1, p, p$ or $p-1$ depending on the case. But this is precisely what we wanted to show, since it forces the degree to be less than or equal to $\frac{p-1}{2}$ in the first three cases, and strictly less in the fourth.

Lastly, $-f_{\theta^p}T$ is linear in the c_i in the upper right term, and constant in the rest, so doesn't cause any problems for $p \geq 3$. $\square$

We can immediately conclude:

Corollary 7.2. *The subscheme of $U_{\mathrm{ns}} \times \mathbb{A}^3$ giving connections with p-curvature 0 is finite over U_{ns}.*

We are now ready to put together previous results to finish the proof of our main theorem in the case of characteristic 7:

Proof of Theorem 1.2, $p = 7$ case. We simply apply our finiteness result to our explicit example from Lemma 6.4. We calculated that $F_{2,7}$ has dimension 5, so by properness the local ring of at least one point in our example has dimension 5, hence they all do. By the reducedness of our example, all its points are unramified over the base, and by finiteness, we conclude that on an open subset of the base containing our chosen point, $F_{2,7}$ is finite and unramified, and everywhere 5-dimensional. Then, by the regularity of the base, we find that over this open set, $F_{2,7}$ must be regular, hence flat, hence étale, so we conclude the desired statement for a general curve in characteristic 7. $\square$

8. Connections and Nodes

This section and the next draw heavily on the results and ideas of Sections 2 and 3 of [13].

In this section, we discuss connections on nodal curves, and classify them in terms of gluings of connections on the normalization. For the sake of simplicity and generality, we follow Mochizuki's argument for the gluing, with the only difference being that because we are not working with projective bundles, we must rigidify our situation by specifying glued line sub-bundles $\mathscr{L}$, as in Proposition 8.10.

Let C be a proper nodal curve, and $\mathscr{E}$ a vector bundle on C. We begin by fixing some terminology:

Definition 8.1. A **logarithmic connection** on $\mathscr{E}$ is a k-linear map $\nabla : \mathscr{E} \to \mathscr{E} \otimes \omega_C$, where ω_C is the dualizing sheaf on C, and ∇ satisfies the Liebnitz rule induced by the canonical map $\Omega^1_C \to \omega_C$. Given a reduced divisor D supported on the smooth locus of C, a **D-logarithmic connection** on $\mathscr{E}$ is a k-linear map $\nabla : \mathscr{E} \to \mathscr{E} \otimes \omega_C(D)$ satisfying the Liebnitz rule.

We note that with the exception of the Cartier isomorphism, all the properties of connections and p-curvature which we have used still hold if one replaces Ω^1_C by ω_C (and in particular, the sheaf of derivations by $\omega_C^\vee$) throughout. We summarize as follows.

Proposition 8.2. *All statements on induced connections for operations of vector bundles, and the basic properties of the p-curvature map of Proposition 2.8, hold in the case of logarithmic connections on nodal curves, with ω_C in place of Ω^1_C. One still has a canonical connection on a Frobenius pullback with vanishing p-curvature whose kernel recovers the original sheaf.*

Although it is true that taking the kernel of the canonical connection of a Frobenius pullback still recovers the original sheaf on $C^{(p)}$ when C is singular, the Cartier isomorphism fails because given a logarithmic connection with vanishing p-curvature on C, the Frobenius pullback of the kernel will not in general map surjectively onto the original sheaf at the singularities of C.

Notation 8.3. Let $\tilde{C}$ be the normalization of C, and $\tilde{\mathscr{E}}$ the pullback of $\mathscr{E}$ to $\tilde{C}$. Given a logarithmic connection ∇ on $\mathscr{E}$, we get a D_C-logarithmic connection $\tilde{\nabla}$ on $\tilde{\mathscr{E}}$, where D_C is the divisor of points lying above the nodes of C.

We want a complete description of connections on $\tilde{\mathscr{E}}$ arising this way, and a correspondence between these and connections on $\mathscr{E}$. We claim:

Proposition 8.4. *Logarithmic connections ∇ on $\mathscr{E}$ are equivalent to connections on $\tilde{\mathscr{E}}$ having simple poles at the points $P_1, Q_1, \dots, P_\delta, Q_\delta$ lying above the nodes of C, and such that under the gluing maps $G_i : \tilde{\mathscr{E}}|_{P_i} \to \tilde{\mathscr{E}}|_{Q_i}$ giving $\mathscr{E}$, for each i we have $\operatorname{Res}_{P_i}(\nabla) = -G_i^{-1} \circ \operatorname{Res}_{Q_i}(\nabla) \circ G_i$. The properties of having trivial determinant and vanishing p-curvature are preserved under this correspondence.*

Proof. The main assertion follows easily from [1, Thm. 5.2.3] together with the remark [1, p. 226] for nodal curves, which together state that sections of ω_C correspond to sections of $\Omega^1_{\tilde{C}}(D_C)$ with residues at the pair of points above any given node adding to zero.

Since vanishing p-curvature can be verified on open sets, and the normalization map is an isomorphism away from the nodes, it is clear that logarithmic connections with vanishing p-curvature on C will correspond to logarithmic connections with vanishing p-curvature on $\tilde{C}$. The same argument also works for trivial determinant. □

We can in particular conclude:

Corollary 8.5. *Let $\mathscr{L}$ be a line bundle on C. Then $\mathscr{L}$ can have a logarithmic connection ∇ with vanishing p-curvature only if $p | \deg \tilde{\mathscr{L}}$.*

Proof. Applying the previous proposition, if we pull back to $\tilde{\nabla}$ on $\tilde{\mathscr{L}}$ we find that the residues of $\tilde{\nabla}$ come in additive inverse pairs mod p. We obviously have $p|\mathscr{F}^*(\tilde{\mathscr{L}}^{\tilde{\nabla}})$, and then by [13, Cor. 2.11] we have that the determinant of the inclusion map $\mathscr{F}^*(\tilde{\mathscr{L}}^{\tilde{\nabla}}) \hookrightarrow \tilde{\mathscr{L}}$ has total order equal to the sum of the residues mod p, which is zero, so we conclude that $\deg \tilde{\mathscr{L}}$ must also vanish mod p, as asserted. □

We now restrict to the situation:

Situation 8.6. Suppose that $\mathscr{E}$ has rank 2 and trivial determinant, and we have fixed an exact sequence

$$0 \to \mathscr{L} \to \mathscr{E} \to \mathscr{L}^{-1} \to 0.$$

The same statements then hold for $\tilde{\mathscr{E}}$.

We introduce some terminology in this situation:

Definition 8.7. Given a logarithmic connection ∇ on $\mathscr{E}$ (resp., a D-logarithmic connection ∇ on $\tilde{\mathscr{E}}$), the **Kodaira-Spencer map** associated to ∇ and a sub-line-bundle $\mathscr{L}$ (resp., $\tilde{\mathscr{L}}$) is the natural map $\mathscr{L} \to \mathscr{L}^{-1} \otimes \omega_C$ (respectively, $\tilde{\mathscr{L}} \to$

$\tilde{\mathscr{L}}^{-1} \otimes \Omega^1_{\tilde{C}}(D))$ obtained by composing the inclusion map, ∇, and the quotient map. One verifies directly that this is a linear map.

In the case that $\mathscr{E}$ (resp., $\tilde{\mathscr{E}}$) is unstable, we will refer to the Kodaira-Spencer map of ∇ to mean the map associated to ∇ and its destabilizing line bundle.

Recall that by Lemma 2.5, the destabilizing line bundle is unique, so the last part of the definition is justified. Note that with this terminology, Joshi and Xia's proof of 2.3 boils down to the statement that the Frobenius-pullback of a Frobenius-unstable bundle necessarily has a connection such that the Kodaira-Spencer map of the destabilizing line bundle is an isomorphism. It should perhaps therefore not be surprising that we will consider connections for which the Kodaira-Spencer is an isomorphism. We note:

Lemma 8.8. *Suppose that the arithetmic genus of C (resp., the genus of $\tilde{C}$ plus $\frac{\deg D}{2}$) is greater than or equal to $3/2$; that is to say, we are in the "hyperbolic" case. Then if the Kodaira-Spencer map associated to $(\nabla, \mathscr{L})$ is an isomorphism for any ∇, then $\mathscr{L}$ is a destabilizing line bundle for $\mathscr{E}$ (resp., $\tilde{\mathscr{E}}$), and is thus uniquely determined even independent of ∇.*

Proof. The Kodaira-Spencer isomorphism gives $\mathscr{L}^{\otimes 2} \cong \omega_C$ (resp., $\mathscr{L}^{\otimes 2} \cong \Omega^1_{\tilde{C}}(D)$), which from the hypotheses has positive degree. □

One can approach the issue of gluing connections from two perspectives: either fixing the glued bundle $\mathscr{E}$ on C, and exploring which connections on $\tilde{\mathscr{E}}$ will glue to yield connections on $\mathscr{E}$, or allowing the gluing of $\mathscr{E}$ itself to vary as well. The author had originally intended to use the first approach, since we ultimately wish to classify the connections on a particular unstable bundle on a nodal curve. However, the second approach, pursued by Mochizuki [11, p. 118], offers a more transparent view of the more general setting, and ultimately yields a cleaner argument even for our specific application. As such, we now fix $\tilde{\mathscr{E}}$ on $\tilde{C}$, but do not assume a fixed gluing $\mathscr{E}$ on C. That is to say:

Situation 8.9. Fix $\tilde{\mathscr{E}}$ of rank 2 and trivial determinant, together with an exact sequence

$$0 \to \tilde{\mathscr{L}} \to \tilde{\mathscr{E}} \to \tilde{\mathscr{L}}^{-1} \to 0.$$

The main statement on gluing is:

Proposition 8.10. *In Situation 8.9, let $\tilde{\nabla}$ be a D_C-logarithmic connection on $\tilde{C}$ with trivial determinant and vanishing p-curvature, such that the Kodaira-Spencer map associated to $\tilde{\mathscr{L}}$ is an isomorphism. Further suppose that the e_1, e_2 of* [13, Cor. 2.10] *match one another (up to permutation) for pairs of points lying above given nodes of C. Then if one fixes a gluing $\mathscr{L}$ of $\tilde{\mathscr{L}}$ with $\mathscr{L}^{\otimes 2} \cong \omega_C$, there is a unique gluing of $(\tilde{\mathscr{E}}, \tilde{\nabla})$ to a pair $(\mathscr{E}, \nabla)$ on C, such that one obtains a sequence*

$$0 \to \mathscr{L} \to \mathscr{E} \to \mathscr{L}^{-1} \to 0,$$

and the resulting $(\mathscr{E}, \nabla)$ will also have Kodaira-Spencer map an isomorphism. If C has arithmetic genus at least 2, transport equivalence is preserved under this correspondence.

Proof. We first claim that the condition that the Kodaira-Spencer map for $\tilde{\mathscr{L}}$ be an isomorphism implies that for any $P \in \{P_i, Q_i\}$, $\tilde{\mathscr{L}}|_P$ is not contained in an

eigenspace of $\operatorname{Res}_P \tilde{\nabla}$, and that the eigenvalues are both non-zero. But due to the triviality of the determinant, the sum of the eigenvalues is zero, so because the residue matrices are diagonalizable (see [13, Cor. 2.11]), the latter assertion is actually a consequence of the former. Now, considering the definition of the Kodaira-Spencer map $\tilde{\mathscr{L}} \to \tilde{\mathscr{L}}^{-1} \otimes \Omega^1_{\tilde{C}}(D_C)$, if we restrict to P we get a map which is clearly equal to zero if and only if $\nabla(\tilde{\mathscr{L}})|_P \subset \tilde{\mathscr{L}} \otimes \Omega^1_{\tilde{C}}|_P$, which is the case precisely when $\tilde{\mathscr{L}}|_P$ is contained in an eigenspace of $\operatorname{Res}_P \tilde{\nabla}$, as desired.

Given this, for each pair P_i, Q_i, Proposition 8.4 and our hypothesis on the matching eigenvalues of the residue matrices at P_i, Q_i imply that in order to glue the connection, it is necessary and sufficient to map eigenspaces of opposing sign to each other. To glue $\tilde{\mathscr{L}}$, we also map its image at P_i to its image at Q_i. We thus see that the two eigenspaces of $\operatorname{Res}_{P_i} \tilde{\nabla}$ and $\operatorname{Res}_{Q_i} \tilde{\nabla}$ and the images of $\tilde{\mathscr{L}}$ form a set of three one-dimensional subspaces which must be matched under G_i, and it is easy to see that this determines G_i up to scaling. But finally, scaling of G_i is equivalent to scaling the induced gluing map on $\tilde{\mathscr{L}}$, which is precisely what determines the isomorphism class of the glued $\mathscr{L}$; thus, $\mathscr{L}$ may be specified arbitrarily, and given a choice of $\mathscr{L}$, the G_i and hence the pair $(\mathscr{E}, \nabla)$ are uniquely determined, as desired. Lastly, we observe that since the Kodaira-Spencer map gives an isomorphism $\mathscr{L} \otimes (\mathscr{E}/\mathscr{L})^{-1} \cong \omega_C$, the hypothesis that $\mathscr{L}^{\otimes 2} \cong \omega_C$ is equivalent to the condition that the glued $\mathscr{E}$ have trivial determinant.

Considering transport, it is trivial that if two connections on $\mathscr{E}$ are transport-equivalent, then their pullbacks to $\tilde{\mathscr{E}}$ are, and for the converse, the uniqueness of the gluing makes it clear that if two connections $\tilde{\nabla}$ and $\tilde{\nabla}'$ on $\tilde{\mathscr{E}}$ are transport-equivalent under an automorphism φ of $\tilde{\mathscr{E}}$, then φ naturally gives an isomorphism of the two gluings $\mathscr{E}$ and $\mathscr{E}'$ which takes ∇ to ∇'. Finally, the hypothesis that the arithmetic genus of C is at least 2 implies that $\mathscr{L}$ and $\tilde{\mathscr{L}}$ are uniquely determined as the destabilizing sub-bundles of $\mathscr{E}$ and $\tilde{\mathscr{E}}$, so there is no concern that they might change under transport. □

Putting together the previous propositions, we finally conclude:

Corollary 8.11. *Let $\tilde{\mathscr{E}}$ be a vector bundle on $\tilde{C}$ of rank 2, with the arithmetic genus of C being at least 2, and suppose there exists an exact sequence*

$$0 \to \tilde{\mathscr{L}} \to \tilde{\mathscr{E}} \to \tilde{\mathscr{L}}^{-1} \to 0.$$

Fix a gluing of $\tilde{\mathscr{L}}$ to a line bundle $\mathscr{L}$ on C satisfying $\mathscr{L}^2 \cong \omega_C$. Then there exists a bijective equivalence between transport-equivalence classes of D_C-logarithmic connections $\tilde{\nabla}$ on $\tilde{\mathscr{E}}$ with trivial determinant and vanishing p-curvature, the eigenvalues of the residues of $\tilde{\nabla}$ matching at the pairs of points above each node, and having the Kodaira-Spencer map an isomorphism on one side, and on the other side, pairs $(\mathscr{E}, \nabla)$ of gluings of $\mathscr{E}$ preserving an exact sequence

$$0 \to \mathscr{L} \to \mathscr{E} \to \mathscr{L}^{-1} \to 0,$$

together with logarithmic connections ∇ on $\mathscr{E}$ with vanishing p-curvature and trivial determinant and having the Kodaira-Spencer map an isomorphism, up to isomorphism and transport equivalence.

Further, this correspondence holds for first-order infinitesmal deformations.

Proof. We can immediately conclude the statement over a field from our previous propositions. For first-order deformations, the same arguments will go through,

with the aid of the following facts: first and most substantively, it follows from [13, Cor. 3.6] that the residue matrices on $\tilde{C}$ will still be diagonalizable over $k[\epsilon]/\epsilon^2$, with the eigenvalues e_i the same as for the connection being deformed. Next, since we are simply taking a base change of our original situation over k, the general gluing description given by Proposition 8.4 still holds for formal reasons. Finally, one can easily verify that even over an arbitrary ring, it is still the case that an automorphism of a rank two free module is determined uniquely by sending any three pairwise independent lines to any other three. We therefore conclude the desired statement for first-order deformations as well. □

9. Deforming to a Smooth Curve

The ultimate goal of this section is to prove that the connections we are interested in can always be smoothed from a general irreducible rational nodal curve, which together with the finiteness result of Section 7 and the main results of [15], [13], will allow us to finish the proof of the characteristic-independent portion of Theorem 1.2. We begin with some general observations on when the space of connections with vanishing p-curvature is smooth over a given deformation of the curve and vector bundle. We then make a key dimension computation using the techniques of [13] and of the previous section, once again following arguments of Mochizuki [11, Cor. II.2.5, p. 150] rather than the original approach of the author, for the sake simplicity and generality.

Situation 9.1. We suppose that C_0 is an irreducible, rational proper curve with two nodes, $\tilde{C}_0 \cong \mathbb{P}^1$ its normalization, with P_1, Q_1, P_2, Q_2 being the points lying above the two nodes. We let $\mathscr{E}_0$ be the vector bundle described by Situation 5.1, and ∇_0 a logarithmic connection on $\mathscr{E}_0$ with trivial determinant and vanishing p-curvature.

By Proposition 8.2, p-curvature gives an algebraic morphism $\psi_p : H^0(\mathcal{E}nd^0(\mathscr{E}_0)\otimes\omega_{C_0}) \to H^0(\mathcal{E}nd^0(\mathscr{E}_0)\otimes F^*\omega_{C_0^{(p)}})$ such that for $\varphi \in H^0(\mathcal{E}nd^0(\mathscr{E}_0)\otimes\omega_{C_0})$, $\psi_p(\nabla_0+\varphi)$ in fact lies in $H^0(\mathcal{E}nd^0(\mathscr{E}_0)\otimes F^*\omega_{C_0^{(p)}})^{(\nabla_0+\varphi)^{\mathrm{ind}}}$. Now, we first claim:

Lemma 9.2. *If ∇_0 has vanishing p-curvature, the differential of ψ_p at 0 gives a linear map*

$$d\psi_p : H^0(\mathcal{E}nd^0(\mathscr{E}_0)\otimes\omega_{C_0}) \to H^0(\mathcal{E}nd^0(\mathscr{E}_0)\otimes F^*\omega_{C_0^{(p)}})^{\nabla_0^{\mathrm{ind}}}.$$

Proof. We simply consider the induced map on first-order deformations of ∇_0. Denoting for the moment by C_1, $\mathscr{E}_1$ the base change of C_0, $\mathscr{E}_0$ to $k[\epsilon]/(\epsilon^2)$, suppose that $\varphi \in \epsilon H^0(\mathcal{E}nd^0(\mathscr{E}_1)\otimes\omega_{C_1}) \cong H^0(\mathcal{E}nd^0(\mathscr{E}_0)\otimes\omega_{C_0})$, and consider $\nabla_0+\varphi$. Since ∇_0 has vanishing p-curvature, the image under ψ_p is in $\epsilon H^0(\mathcal{E}nd^0(\mathscr{E}_1)\otimes F^*\omega_{C_1^{(p)}})^{(\nabla_0+\epsilon\varphi)^{\mathrm{ind}}}$, which is naturally isomorphic to $H^0(\mathcal{E}nd^0(\mathscr{E}_0)\otimes F^*\omega_{C_0^{(p)}})^{\nabla_0^{\mathrm{ind}}}$, giving the desired result. □

Our main assertion is:

Proposition 9.3. *If the map $d\psi_p$ of the previous lemma is surjective, then given a deformation C of C_0 and $\mathscr{E}$ of $\mathscr{E}_0$ on C, such that the functor of connections on $\mathscr{E}$ with trivial determinant is formally smooth at ∇_0, then the functor of connections on $\mathscr{E}$ with trivial determinant and vanishing p-curvature is formally smooth at ∇_0.*

Proof. By hypothesis, there is no obstruction to deforming ∇_0 as a connection with trivial determinant. Following [17, Def. 1.2, Rem. 2.3], we say that a map $B \twoheadrightarrow A$ of local Artin rings over the base ring of our deformation and having residue field k is a **small extension** if the kernel is a principal ideal (ϵ) with $(\epsilon)\mathfrak{m}_B = 0$; it follows then that $\epsilon B \subset B$ is isomorphic to k. To verify (formal) smoothness, by virtue of [18, Prop. 17.14.2] it is easily checked inductively that it is enough to check on small extensions. We show therefore that for such a small extension, when $d\psi_p$ is surjective there is no obstruction to lifting a deformation of ∇_0 over A to a deformation over B, even with the addition of the vanishing p-curvature hypothesis. Let $C_B, \mathscr{E}_B$ be the given deformations over B of $C_0, \mathscr{E}_0$ respectively, with $C_A, \mathscr{E}_A$ the induced deformations over A, and suppose that ∇_B is a connection on $\mathscr{E}_B$ such that ∇_A has vanishing p-curvature. The main point is that it is straightforward to check that the hypothesis that $\epsilon B \cong k$ implies that $\epsilon H^0(\mathcal{E}nd^0(\mathscr{E}_B) \otimes \omega_{C_B}) \cong H^0(\mathcal{E}nd^0(\mathscr{E}_0) \otimes \omega_{C_0})$, and for any $\varphi \in \epsilon H^0(\mathcal{E}nd^0(\mathscr{E}_B) \otimes \omega_{C_B})$, we have $\epsilon H^0(\mathcal{E}nd^0(\mathscr{E}_B) \otimes F^*\omega_{C_B^{(p)}})^{(\nabla_B+\varphi)^{\text{ind}}} \cong H^0(\mathcal{E}nd^0(\mathscr{E}_0) \otimes F^*\omega_{C_0^{(p)}})^{\nabla_0^{\text{ind}}}$. We want to show that for some choice of $\varphi \in \epsilon H^0(\mathcal{E}nd^0(\mathscr{E}_B) \otimes \omega_{C_B})$, $\nabla_B + \varphi$ has vanishing p-curvature. But as before, since ∇_A has vanishing p-curvature, the image under ψ_p of $\nabla_B + \varphi$ is in $\epsilon H^0(\mathcal{E}nd^0(\mathscr{E}_B) \otimes F^*\omega_{C_B^{(p)}})^{(\nabla_B+\varphi)^{\text{ind}}}$, and under the above isomorphisms, the induced map is equal to $d\psi_p + \frac{1}{\epsilon}\psi_p(\nabla_B)$, where $\frac{1}{\epsilon}$ is simply shorthand for the isomorphism $\epsilon H^0(\mathcal{E}nd^0(\mathscr{E}_B) \otimes F^*\omega_{C_B^{(p)}})^{(\nabla_B+\varphi)^{\text{ind}}} \overset{\sim}{\to} H^0(\mathcal{E}nd^0(\mathscr{E}_0) \otimes F^*\omega_{C_0^{(p)}})^{\nabla_0^{\text{ind}}}$. Hence if $d\psi_p$ is surjective, we can choose φ so that $\nabla_B + \varphi$ has vanishing p-curvature, as desired. $\square$

We observe that in our situation, the normalization $\tilde{\mathscr{E}}_0$ of $\mathscr{E}_0$ is isomorphic to $\mathscr{O}(1) \oplus \mathscr{O}(-1)$: we certainly have $\tilde{\mathscr{L}} \cong \mathscr{O}(1)$, so by Lemma 2.5, $\mathscr{O}(1)$ is the maximal line bundle in $\tilde{\mathscr{E}}_0$, and then the desired splitting follows from [4, Proof of Thm. 1.3.1]. Also, by Proposition 8.4 $\tilde{\nabla}_0$ is a D_{C_0}-logarithmic connection on $\tilde{\mathscr{E}}_0$ with trivial determinant and vanishing p-curvature. For the sake of cleanness and generality, we use Mochizuki's arguments [11, Cor. II.2.5, p. 150] to prove the following.

Proposition 9.4. *If ∇_0 has a non-zero Kodaira-Spencer map, then the space of sections of $\mathcal{E}nd^0(\mathscr{E}_0) \otimes F^*\omega_{C_0^{(p)}}$ horizontal with respect to the connection ∇_0^{ind} induced by ∇_0 on $\mathscr{E}_0$ and ∇^{can} on $F^*\omega_{C_0^{(p)}}$ has dimension* 3.

Proof. The proof proceeds in two parts: we show that $H^1(C_0, (\mathcal{E}nd^0(\mathscr{E}_0) \otimes F^*\omega_{C_0^{(p)}})^{\nabla_0^{\text{ind}}}) = 0$, and then compute the Euler characteristic. Both computations require formal local computations, so we begin by setting out the situation formally locally at a node of C_0. First, note that although taking kernels and tensor products of connections do not commute in general, there is no problem when one connection is obtained as the canonical connection of a Frobenius pullback, so we have $(\mathcal{E}nd^0(\mathscr{E}_0) \otimes \mathscr{F}^*\omega_{C_0^{(p)}})^{\nabla_0^{\text{ind}}} = \mathcal{E}nd^0(\mathscr{E}_0)^{\nabla_0^{\mathcal{E}nd}} \otimes \omega_{C_0^{(p)}}$. Formally locally at the node, C_0 is isomorphic to $k[[x,y]]/(x,y)$; moreover, we claim that if we choose x, y correctly, we can trivialize $\mathscr{E}_0$ so that $\nabla_0^{\mathcal{E}nd}$ has connection matrix $\begin{bmatrix} e(\frac{dx}{x} - \frac{dy}{y}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e(\frac{dy}{y} - \frac{dx}{x}) \end{bmatrix}$ for some e with $0 < e < p$. Indeed, this follows from Proposition 8.4 together with the formal local diagonalizability result of [13, Cor.

2.10] applied to $\tilde{C}_0$, noting that if the residue of ∇_0 has eigenvalues $e', -e'$, then the residue of $\nabla_0^{\mathcal{E}nd}$ has eigenvalues $2e', 0, -2e'$. By the same token, the pullback to the normalization gives connection matrices $\begin{bmatrix} e\frac{dx}{x} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -e\frac{dx}{x} \end{bmatrix}$ and $\begin{bmatrix} -e\frac{dy}{y} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e\frac{dy}{y} \end{bmatrix}$. Finally, we note that the kernel of the connection on C_0 is given over $\mathscr{O}_{C_0^{(p)}}$ by $(x^{p-e}, y^e) \oplus (1) \oplus (x^e, y^{p-e})$, and by $(x^{p-e}) \oplus (1) \oplus (x^e)$ and $(y^e) \oplus (1) \oplus (y^{p-e})$ on the normalization. The formal local calculations of the following paragraphs are justified by the following facts: given a sheaf map, surjectivity, and more generally factoring through a given subsheaf, may be checked after completion; completion commutes with pullback, with taking kernels of connections in characteristic p, and with modding out by torsion over a DVR; finally, completion is well-behaved with respect to pushforward under the normalization map by the theorem on formal functions.

Now, to check that H^1 vanishes, by Grothendieck duality on $C_0^{(p)}$ it suffices to check that $\operatorname{Hom}(\mathcal{E}nd^0(\mathscr{E}_0)^{\nabla_0^{\mathcal{E}nd}} \otimes \omega_{C_0^{(p)}}, \omega_{C_0^{(p)}}) = \operatorname{Hom}(\mathcal{E}nd^0(\mathscr{E}_0)^{\nabla_0^{\mathcal{E}nd}}, \mathscr{O}_{C_0^{(p)}}) = 0$. Although a section of the latter need not come from a map $\mathcal{E}nd^0(\mathscr{E}_0) \to \mathscr{O}_{C_0}$ which is horizontal with respect to $\nabla_0^{\mathcal{E}nd}$, we claim that it does after normalization. We have a natural map $\mathcal{H}om(\mathcal{E}nd^0(\mathscr{E}_0)^{\nabla_0^{\mathcal{E}nd}}, \mathscr{O}_{C_0^{(p)}})|_{\tilde{C}_0} \to \mathcal{H}om(\mathcal{E}nd^0(\mathscr{E}_0)|_{\tilde{C}_0}^{\nabla_0^{\mathcal{E}nd}}, \mathscr{O}_{\tilde{C}_0^{(p)}})$, and a natural inclusion $\mathcal{H}om(\mathcal{E}nd^0(\mathscr{E}_0)|_{\tilde{C}_0}, \mathscr{O}_{\tilde{C}_0})^{\nabla_0^{\mathcal{E}nd}} \hookrightarrow \mathcal{H}om(\mathcal{E}nd^0(\mathscr{E}_0)|_{\tilde{C}_0}^{\nabla_0^{\mathcal{E}nd}}, \mathscr{O}_{\tilde{C}_0^{(p)}})$. These are both isomorphisms away from the points above the nodes, for trivial reasons in the first case, and because of Theorem 2.1 for the second. We want to show that the first map factors through the second. Examining the formal local situation at a node, we first note that if $e_1, e_2 > 0$, any map from (x^{e_1}, y^{e_2}) to $\mathscr{O}_{C_0^{(p)}}$ necessarily vanishes, and more specifically, sends x^{e_1} and y^{e_2} to positive (pth) powers of x and y respectively. It is thus clear that give a map $\mathcal{E}nd^0(\mathscr{E}_0)^{\nabla_0^{\mathcal{E}nd}} \to \mathscr{O}_{C_0^{(p)}}$, after normalization we can divide through to get a map formally locally $\mathcal{E}nd^0(\mathscr{E}_0)|_{\tilde{C}_0} \to \mathscr{O}_{\tilde{C}_0}$ which commutes with the induced connection, completing the proof of the claim. Next, we claim that such a map must be 0. Indeed, if we consider the line sub-bundle $\mathscr{L}^0 \subset \mathcal{E}nd^0(\mathscr{E}_0)$ which sends $\mathscr{L} \subset \mathscr{E}_0$ to 0, we see that it is isomorphic to $\mathscr{L}^{\otimes 2}$, and is not horizontal for $\nabla_0^{\mathcal{E}nd}$, since $\mathscr{L}$ is not horizontal for ∇_0, and we have the same situation after normalization. But having such a destabilizing line sub-bundle precludes the existence of a horizontal morphism $\mathcal{E}nd^0(\mathscr{E}_0)|_{\tilde{C}_0} \to \mathscr{O}_{\tilde{C}_0}$ by Proposition 2.5, so we conclude the desired vanishing statement.

Thus, it remains to compute the Euler characteristic of $\mathcal{E}nd^0(\mathscr{E}_0)^{\nabla_0^{\mathcal{E}nd}}$. Since we only have two non-zero eigenvalues at each P_i or Q_i, it follows from [13, Cor. 2.11] that the cokernel of $F^*((\mathcal{E}nd^0(\mathscr{E}_0)|_{\tilde{C}_0})^{\nabla_0^{\mathcal{E}nd}}) \to \mathcal{E}nd^0(\mathscr{E}_0)|_{\tilde{C}_0}$ is supported at the P_i, Q_i, with length p at each point. Since $\deg(\mathcal{E}nd^0(\mathscr{E}_0)|_{\tilde{C}_0}) = 0$, we find that $\deg(F^*((\mathcal{E}nd^0(\mathscr{E}_0)|_{\tilde{C}_0})^{\nabla_0^{\mathcal{E}nd}})) = -4p$. Next, we claim that $(\mathcal{E}nd^0(\mathscr{E}_0)|_{\tilde{C}_0})^{\nabla_0^{\mathcal{E}nd}}$ is isomorphic to the quotient of $(\mathcal{E}nd^0(\mathscr{E}_0)^{\nabla_0^{\mathcal{E}nd}})|_{\tilde{C}_0}$ by its torsion, which we denote by $\mathscr{F}$; indeed, we clearly have a morphism from the latter to the former, which is an isomorphism away from the points above the nodes, hence gives an injection since we modded out by torsion. Surjectivity above the nodes is then checked formally locally from our above description, so we have $\deg(\mathscr{F}) = -4$, and

$\chi(\mathscr{F}) = -1$. Finally, if ν denotes the normalization map, we claim that the natural injection $\mathcal{E}nd^0(\mathscr{E}_0)^{\nabla_0^{\mathcal{E}nd}} \hookrightarrow \nu_*\mathscr{F}$ has cokernel of length 1 at each node; again, this is checked formally locally, noting that the cokernel will arise only from the summand at each node on which the connection vanishes. We conclude therefore that $\chi(\mathcal{E}nd^0(\mathscr{E}_0)^{\nabla_0^{\mathcal{E}nd}}) = -3$, so $H^0(\mathcal{E}nd^0(\mathscr{E}_0)^{\nabla_0^{\mathcal{E}nd}} \otimes \omega_{C_0^{(p)}}) = \chi(\mathcal{E}nd^0(\mathscr{E}_0)^{\nabla_0^{\mathcal{E}nd}} \otimes \omega_{C_0^{(p)}}) = 3$, completing the proof of the proposition. □

Finally, we put these results together in our specific situation:

Theorem 9.5. *Let C_0 be a nodal rational curve of genus 2, and $\mathscr{E}_0$ as in Situation 5.1. Let ∇_0 have vanishing p-curvature and trivial determinant, and suppose that ∇_0 has no deformations preserving the p-curvature and not arising from transport. Then the map $d\psi_p$ of Lemma 9.2 is surjective; in particular, given any deformation C of C_0, if $\mathscr{E}$ is the corresponding deformation of $\mathscr{E}_0$, then the space of connections with trivial determinant and vanishing p-curvature on $\mathscr{E}$ is formally smooth at ∇_0.*

Proof. The main point is that by Remark 5.6, the space of transport-equivalence classes of connections with trivial determinant on $\mathscr{E}_0$ or $\mathscr{E}$ is explicitly parametrized by $\mathbb{A}^3$ over the appropriate base. In particular, deformations of ∇_0 as a connection with trivial determinant are unobstructed, and it also follows that the space of first-order deformations of ∇_0 with trivial determinant, modulo those arising from transport, is three-dimensional. By Proposition 9.4, the image space of $d\psi_p$ is three-dimensional. We therefore get surjectivity precisely when transport accounts for the entire kernel, which is to say, when there are no deformations of ∇_0 having vanishing p-curvature and trivial determinant other than those obtained by transport. We can thus apply the previous proposition to conclude smoothness. □

It is now a matter of some simple combinatorics to complete the proof of the characteristic-independent portion of Theorem 1.2.

Proof of Theorem 1.2, $p > 2$ case. By the results of Section 2 it suffices to show that, for the particular $\mathscr{E}$ of Situation 5.1, there are precisely $\frac{1}{24}p(p^2-1)$ transport-equivalence classes of connections with trivial determinant and vanishing p-curvature on $\mathscr{E}$, and that none of these have any non-trivial deformations. We will show that this statement holds in the situation that C is a general rational nodal curve, and then conclude the same result must hold for a general smooth curve.

We observe that even in the situation of a nodal curve, there is a unique extension $\mathscr{E}$ of $\mathscr{L}^{-1}$ by $\mathscr{L}$; indeed, the proof of Proposition 2.6 goes through with ω_C in place of Ω^1_C. We also note that by Corollary 8.5, the argument of Proposition 2.3 still shows that any connection must have its Kodaira-Spencer map be an isomorphism. It then follows from Corollary 8.11 that it suffices to prove the same result for D-logarithmic connections on $\mathscr{O}(1) \oplus \mathscr{O}(-1)$ on $\mathbb{P}^1$ satisfying the hypotheses of [13, Sit. 2.12] and having the Kodaira-Spencer map an isomorphism, where D is made up of four general points on $\mathbb{P}^1$, and the eigenvalues of the residues at the points match in the appropriate pairs. We note that by degree considerations, the Kodaira-Spencer map in this case is always either zero or an isomorphism, so if we fix eigenvalues α_i for each pair (P_i, Q_i), by [13, Thm. 1.1] we find that we are looking for separable rational functions on $\mathbb{P}^1$ of degree $2p-1-2\sum \alpha_i$, and ramified to order at least $p - 2\alpha_i$ at P_i and Q_i (note that the coefficient doubling for the degree is due to our use of a single, matching α_i for both P_i and Q_i). We could use the second formula of [15, Cor. 8.1] to compute the answer directly, but the

first formula yields a more elegant solution. In either case, we are already given the lack of non-trivial deformations, so it suffices to show that the number of maps is correct. The formula gives that for each (α_1, α_2) there are

$$\min\{\{p-2\alpha_i\}_i, \{p-2\alpha_{3-i}\}_i, \{2\alpha_i\}_i, \{2\alpha_{3-i}\}_i\}$$

such maps, which reduces to

$$\min\{\{p-2\alpha_i\}_i, \{2\alpha_i\}_i\}.$$

Rather than summing up over all α_i, as we would with the second formula, we note that the number of maps will also be given by:

$$\sum_{1\leq j\leq (p-1)/2} \#\{(\alpha_1,\alpha_2) : j \leq 2\alpha_i, j \leq p-2\alpha_i\}$$

which then reduces to

$$\sum_{1\leq j\leq (p-1)/2} (\frac{p+1}{2}-j)^2 = \sum_{1\leq j\leq (p-1)/2} j^2 = \sum_{1\leq j\leq (p-1)/2} (2\binom{j}{2}+j)$$

$$= 2\binom{(p+1)/2}{3} + \frac{p+1}{2}\frac{p-1}{4} = \frac{1}{24}(p+1)((p-1)(p-3)+3(p-1)) = \frac{(p+1)(p-1)p}{24},$$

giving the desired result for a general nodal curve.

We can now apply Theorem 9.5 to conclude that since none of our connections on the general nodal curve have non-trivial deformations, the space of connections with trivial determinant and vanishing p-curvature on our chosen bundle over our parameter space of genus 2 curves is formally smooth at each connection on the general nodal curve. Furthermore, by Corollary 7.2 (in light of Remark 5.6), this space of connections is finite, so we conclude that it is finite étale at the general nodal curve, and finite everywhere, which then implies (i) for a general smooth curve, as desired. □

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